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METR 4123, Atmospheric Dynamics
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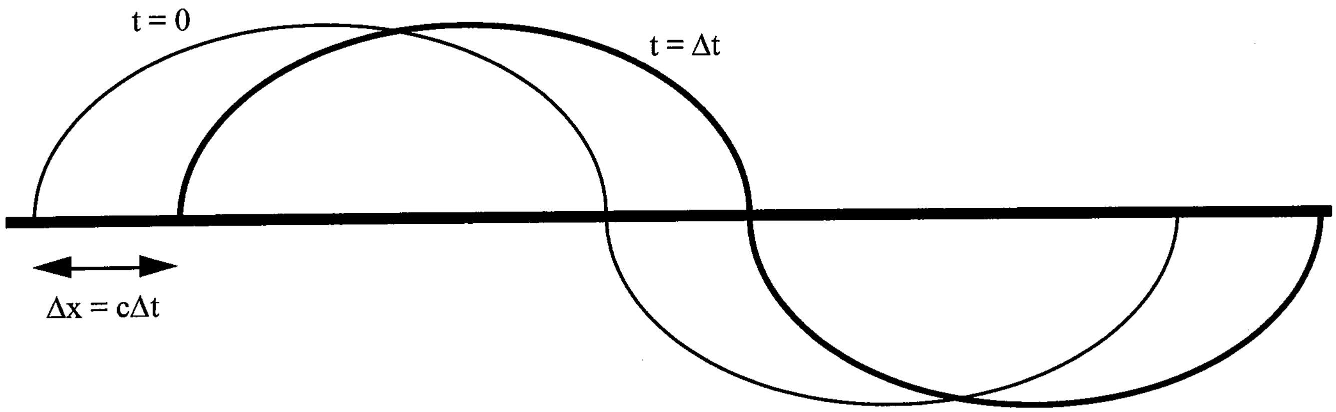
Mathematical Representation / Description of Waves

Consider a wave traveling in the x-direction at a constant speed c and having a profile given by $u(x,0) = f(x)$ (thin curve):



If the wave does not change change shape, then after some time Δt, the wave will have moved a distance $\Delta x = c\Delta t$ (bold curve).

How can we represent the wave at $t = \Delta t$ from a functional point of view? The new wave is simply the old wave moved to the right by an amount $c\Delta t$. Thus, the new position is $x + \Delta x = x + c\Delta t$. From the point of view of the wave,

$$u(x, \Delta t) = f(x - c\Delta t)$$

where $f(x - c\Delta t)$ is the profile in reference to the original location.

The expression $x - ct$ is often called the phase of the wave. If we ride along with a particular point on the wave, then

$$x - ct = \text{constant}.$$

If $c > 0$ (wave moving to the right) and t increases, then we see that x has to increase $\Rightarrow$ phase is advancing to the right. Thus,

$$f(x + ct) \Rightarrow \text{left-moving wave}$$
$$f(x - ct) \Rightarrow \text{right-moving wave.}$$

If the waveform f is given by sines and cosines, it is called a HARMONIC WAVE and has the following functional form:

$$f(x,t) = A\cos(kx - \omega t)$$

$k = \frac{2\pi}{L}$, L = wavelength, k = wavenumber

A = amplitude,

$\omega = kc$, c = phase speed (not particle speed) and ω is the angular frequency (inverse seconds)

Thus, all we need to express a wave is an amplitude and a phase.

Fourier Series

One of the most beautiful concepts of mathematics is the Fourier series--a way to represent an arbitrary (continuous, smooth, periodic) function as a series of sines and cosines:

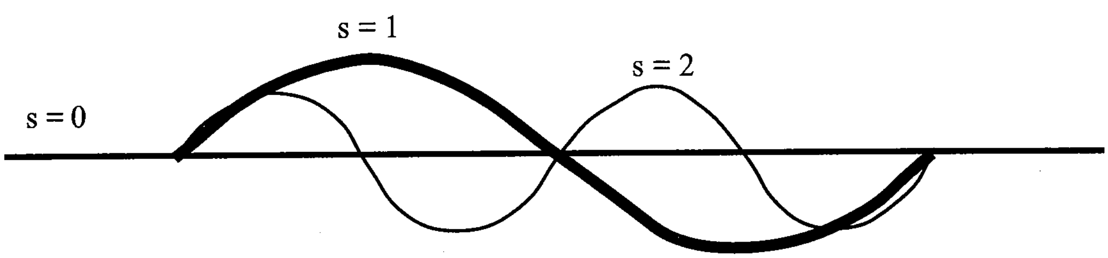
$$f(x) = \sum_{s=1}^{\infty} A_s \sin k_s x + B_s \cos k_s x$$

(linear superposition)

where $k_s = \frac{2\pi s}{L}$ = zonal number

L = distance around a latitude circle

s = integer that equals the number of waves in a latitude circle (see figure)



In practice, s is truncated to some finite number, as in meteorological spectral models. In theory, however, if a sufficient number of terms is used, we obtain an exact representation of the function. Note that this is not the case if the function has true discontinuities, no matter how many terms are used (see discussion next page).

To obtain the unknown Fourier coefficients, A_s and B_s, we multiply the series by $\sin\left(\frac{2\pi nx}{L}\right)$, $n \sim$ integer, and integrate over the wavelength L. Why? Because of the orthogonality property of the trigonometric functions:

$$\int_0^L \sin\left(\frac{2\pi sx}{L}\right)\sin\left(\frac{2\pi nx}{L}\right)dx = \begin{Bmatrix} 0 & s \neq n \\ L/2 & s = n \end{Bmatrix}$$

$$\Rightarrow A_S = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi sn}{L}\right)dx$$

Doing the same for cosine gives an expression for B_s:

$$B_s = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi sx}{L}\right)dx$$

A_s and B_s are called the <u>FOURIER COEFFICIENTS</u>, and

$$f_s(x) = A_s \sin(k_s x) + B\cos(k_s x)$$

is the s^{th} <u>HARMONIC</u>. If the flow behaves like waves, we may need only $s = 1$ or $s = 1,2$ to capture most of the structure. *<u>NOTE</u>:* Given observations of some field variable, such as the zonal wind u, we can write it as a Fourier series!

If one tries to represent a discontinuity using a Fourier series (see figure on the next page), oscillations known as the <u>Gibbs phenomenon</u> occur even if the <u>spectral resolution</u> is continually increased to infinity. Why? Because a discontinuous function cannot be represented <u>exactly</u> by a Fourier series.

To avoid the cumbersome practice of writing sine and cosine, we often use a shorthand notation for the Fourier series

$$f_s(x) = Re\,\{c_s\, e^{(ik_s x)}\}$$

$$= R_e\,\{c_s \cos k_s x + i c_s \sin k_s x\}$$

where Re denotes the real part of the expression contained in {brackets}. Once can show that

$$B_s = R_e\,\{c_s\}$$
$$A_s = -I_m\,\{c_s\}$$

We'll most often use, for example,

$$u = \tilde{u}\, e^{i(kx-\omega t)}.$$

where $\omega = \omega_R + i\omega_I$ is the complex frequency. Substituting, we have

$$u = \tilde{u}\, e^{i(kx-\omega_R t)}\, e^{\omega_I t}$$

The first exponential function describes the wave structure and propagation (sines and cosines) while the second denotes amplification or damping. Note that there is damping if $\omega_I < 0$, no change if $\omega_I = 0$, and growth if $\omega_I > 0$. As a result, a wave having a complex frequency (ω) or phase speed $(c = \omega/k)$ may grow or be damped. Note that we only use the <u>real part</u> of the expression for u, i.e.,

$$u = R_e\left\{\tilde{u}\, e^{i(k_x-\omega t)}\right\}$$

because the real part is the only one that has physical significance. The imaginary part is present only as a notational convenience.