Lifting of Ambient Air by Density Currents in Sheared Environments

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ABSTRACT

Two aspects of vorticity associated with cold pools are addressed. First, tilting of horizontal vortex tubes by the updraft at a gust front has been proposed as a means of getting near-ground rotation and hence a tornado. Theory and a numerical simulation are used to show that this mechanism will not work because warm air parcels approaching the gust front decelerate in strong adverse pressure gradient. The near-surface horizontal vorticity available for upward tilting is greatly reduced by horizontal compression before it is tilted. Consequently, uplifting of vortex tubes produces little vertical vorticity near the ground.

Second, it is shown that the baroclinic vorticity generated at the leading edge of the cold pool is transported rearward in the vortex sheet along the interface between cold and warm air, and the barotropic vorticity associated with environmental shear is conserved along streamlines. Warm parcels away from the interface do not acquire baroclinic vorticity to offset their barotropic vorticity, as assumed in a theory for long-lived squall lines. The vortex sheet has a far-field effect on the circulation in the warm air. A steady-state vortex method is used to propose why there is a steady noncirculating density current only when a lid is present and at a specific height.

1. Introduction

The tilting of storm-relative environmental streamwise vorticity by an updraft explains updraft rotation and mesocyclone formation aloft (e.g., Lilly 1982; Davies-Jones 1984). However, in an environment devoid of vertical vorticity at the surface, rotation next to the ground does not seem to develop without a downdraft present nearby (Davies-Jones 1982a,b, 2008). In this paper we address whether, without a downdraft playing a role, environmental vortex lines can be tilted abruptly upward by a gust front, leading to strong vertical vorticity very close to the ground that can be stretched into a tornadic vortex. This process is called hereafter the gust-front mechanism. In this regard, Simpson (1982) proposed that a waterspout may form as a result of a steep density current gust front scooping up a bundle of horizontal vortex tubes from the sea surface and connecting these tubes to a mesocyclone that has extended downward to the base of an overlying convective cloud (Fig. 1). We should mention, however, that this hypothesis did not appear in the later stage of this paper (Simpson et al. 1986). Davies-Jones (1982a,b) pointed out that vorticity tilted by an updraft alone acquires a vertical component only as it rises away from the surface. At the same time as it is being produced, vertical vorticity is being advected away from the ground. Thus, it seems that in the absence of a downdraft vertical vorticity can be present very near to the ground only if vortex lines near the surface are turned abruptly upward by intense gradients of upward velocity. Adlerman et al. (1999, p. 2045) claimed that this is highly improbable without either a strong vortex being present already at low levels to provide strong upward pressure-gradient forces or a gust front. Davies-Jones et al. (2001) attempted to rule out the gust-front mechanism by pointing out that low-level mesocyclones form in numerical simulations that do not have the fine grid spacing necessary to resolve the abrupt upward turning of vortex lines. This argument is inductive. Below, we use theory, supported by a numerical simulation, to provide physical reasons why, even in

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extreme environmental shear, the gust-front mechanism fails to produce significant vertical vorticity in the lowest few hundred meters of the atmosphere.

A related issue is the regeneration of convection by lifting of inflowing air at the leading edge of cold pools (Liu and Moncrieff 1996; Moncrieff and Liu 1999). Xu (1992) and Xu and Moncrieff (1994) concentrated on obtaining steady inviscid solutions for two-fluid density currents and demonstrated how the solutions were controlled largely by global balances. They also proved that the interface still intersects the ground at a 60° angle even when shear, circulations in the cold pool, and a top boundary are present. Their other results are affected heavily by a rigid lid, which they require to obtain steady flows. The density-current height is a considerable fraction of the domain height, so the lid definitely restricts the amount of lift.

Rotunno et al. (1988, hereafter RKW) and Weisman and Rotunno (2004, hereafter WR) concentrated on the vorticity dynamics of the interaction of the cold pool with the shear flow locally near the gust front. According to them, deep lifting at the leading edge of cold pools occurs when the circulations associated with the cold pool and the low-level environmental shear balance one another. RKW use a control-volume analysis of vorticity generation and fluxes to derive an optimum condition for upward displacement of air parcels in steady inviscid flow. They equate a vertically oriented jet with zero net efflux of vorticity through the top boundary of the control volume, which is perhaps a questionable assumption given their simulation results (note that $\partial w/\partial z \neq 0$ where the flow is vertical in their Fig. 20b and $\partial w/\partial x$ does not contribute to the net flux because $w = 0$ at the corners of the control volume). The approximate control-volume analysis is in fact unnecessary because the vorticity equation has an integral if the flow is assumed to be isentropic as well as inviscid and steady. From this, we deduce the strength of the interfacial vortex sheet in a two-fluid flow (Xu 1992; Xu and Moncrieff 1994), and show that the properties of the baroclinic and barotropic vorticity are inconsistent with the RKW theory. We then model the interfacial vortex sheet using discrete vortices to demonstrate why a steady solution with a noncirculating cold pool requires an upper boundary at a specific height.

2. Vorticity in two-dimensional, three-directional flow

We start by considering vorticity in a simple flow that reveals how horizontal vortex lines are tilted upward at an “obstacle” such as a gust front. This flow is two dimensional ($\partial / \partial y = 0$), inviscid, and isentropic. The flow is three directional to provide a component of horizontal vorticity that can be tilted. The y momentum in this slab-symmetric flow is equivalent to the angular momentum in an axisymmetric flow. Coriolis forces are omitted to eliminate ambient vertical vorticity. The momentum, mass continuity, and entropy equations are

$$\frac{dv}{dt} = -c_p \theta V \pi - V(gz), \quad (1)$$
$$\frac{d\alpha}{dt} = \alpha V \cdot v, \quad \text{and} \quad (2)$$
$$\frac{d\theta}{dt} = 0, \quad (3)$$

where the position vector is $x = xi + yj + zk$; the velocity vector is $v = ui + vj + wk$; $i$, $j$, and $k$ are the unit eastward, northward, and upward vectors; $t$ is time; $\alpha$ is specific volume; $p$ is pressure; $\pi = (p/p_0)^{\kappa}$ is the non-dimensional pressure; $T$ is temperature; $\theta = T/\pi$ is potential temperature; $V = (\partial \phi / \partial x, 0, \partial \phi / \partial z)$; and $d/dt = \partial \phi / \partial t + u \partial \phi / \partial x + w \partial \phi / \partial z$. The constants are $g$, the acceleration due to gravity; $R$, the gas constant for dry air; $c_p = (7R/2)$, the specific heat of dry air at constant pressure; $\kappa = R/c_p$ ($=\gamma$); and $p_0$, a standard pressure (1000 mb). The equation set is closed by the ideal gas law

$$\alpha = \frac{RT}{p} = \frac{R\theta \pi}{p_0 \pi^{\kappa}\gamma R}. \quad (4)$$

For slab-symmetric flow, the vorticity is defined by

$$\omega = \xi i + \eta j + \zeta k = \left( -\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} \right). \quad (5)$$
From (1) and (2), the vorticity equation is
\[
\frac{1}{\alpha} \frac{d(\alpha \omega)}{dt} = (\omega \cdot \nabla) v + c_p \nabla \times \nabla \theta = \frac{\partial (u, v)}{\partial (z, x)} + \frac{\partial (\pi, c_p \theta)}{\partial (z, x)} + \frac{\partial (w, v)}{\partial (z, x)} k, \tag{6}
\]
where the Jacobian \([\partial (u, v)/\partial (z, x)] = [(\partial u/\partial z)(\partial v/\partial x)] - [(\partial u/\partial x)(\partial v/\partial z)]\), etc.

We obtain the integral of (6) by introducing Lagrangian coordinates \((X, Y, Z, \tau)\), where \(\tau = t, \tau_0\) is the initial time, and \((X, Y, Z, \tau_0)\) are the initial coordinates of the parcel currently at \((x, y, z, t)\). The specific volume and velocity of this parcel at \(\tau_0\) are \(\alpha_0\) and \((u_0, v_0, w_0)\), respectively. The parcel’s initial vorticity is
\[
\omega_0 = (\xi_0, \eta_0, \zeta_0) = \left( \frac{\partial u_0}{\partial Z}, \frac{\partial u_0}{\partial X}, \frac{\partial w_0}{\partial X} \right). \tag{7}
\]
Note from \(\mathbf{j} \cdot (1)\) and (3) that \(v\) and \(\theta\) are conserved following a parcel; that is,
\[
v(x, y, z, t) = v_0(X, Y, Z, \tau_0) \tag{8}
\]
\[
\theta(x, y, z, t) = \theta_0(X, Y, Z, \tau_0) \tag{9}
\]
Since \(v\) does not appear in any other equation in the set, it is a passive scalar. In other words, the flow in the \(x-z\) plane is unaffected by \(v\). The Lagrangian continuity equation for the symmetric flow is
\[
\frac{\partial (X, Z)}{\partial (z, x)} = \alpha_0 \quad \text{or} \quad \frac{\partial (X, Z)}{\partial z} = \frac{\partial \alpha_0}{\partial z} \tag{9}
\]
From (6), (8), and (9), the vorticity equation in Lagrangian coordinates is
\[
\frac{1}{\alpha_0} \frac{\partial (\alpha \omega)}{\partial \tau} - \frac{\partial (u, v_0)}{\partial (Z, X)} - \frac{\partial (w, w_0)}{\partial (Z, X)} k = \frac{\partial (\pi, c_p \theta_0)}{\partial (Z, X)} j = \frac{\partial \pi - f(\theta_0)}{\partial (Z, X)} j. \tag{10}
\]
The function \(f(\theta_0)\) is superfluous for steady flows, but necessary for steady flows (section 3). After defining \(\Pi = \int_0^\tau \left( \pi (\tau) - f(\theta_0) \right) d\tau\), which is the integral of \(\pi\) following a parcel, and using the identities \(\partial x/\partial \tau = u, \partial z/\partial \tau = w, \partial v_0/\partial \tau = 0, \text{and} \partial \theta_0/\partial \tau = 0\), we can rewrite (10) as
\[
\frac{\partial}{\partial \tau} \left[ \frac{\alpha \omega}{\alpha_0} - \frac{\partial (x, v_0)}{\partial (Z, X)} - \frac{\partial (\Pi, c_p \theta_0)}{\partial (Z, X)} j - \frac{\partial (z, w_0)}{\partial (Z, X)} k \right] = 0. \tag{11}
\]
Integration, application of the initial condition (7), and reuse of (9) yields the vorticity formula
\[
\omega = \left[ -\frac{\partial v_0}{\partial z} \frac{\alpha}{\alpha_0} \eta_0 + \frac{\partial (\Pi, c_p \theta_0)}{\partial (Z, X)} \frac{\partial v_0}{\partial x} \right]. \tag{12}
\]
Note from (12) that \(v_0\) serves as a “streamfunction” for the \((\xi, 0, \zeta)\) vector field. Thus, the vortex lines lie in surfaces of constant \(v_0\).

In simulations of convective storms, the initial state often is an unperturbed horizontally homogeneous environment. In such cases, \(v_0 = v_0(Z)\) and there is no initial vertical vorticity. From (12) it is evident that
\[
\omega \cdot \nabla Z = \frac{\partial (Z, v_0(Z))}{\partial (z, x)} = 0, \tag{13}
\]
so the vortex lines lie in constant-\(Z\) surfaces (which coincide here with constant-\(v_0\) surfaces, and the isentropic surfaces as well if the environment is stably stratified). It follows from (13) that
\[
\xi = \left( \frac{\partial z}{\partial x} \right)_Z \zeta. \tag{14}
\]
This suggests that significant vertical vorticity could be produced near the ground by abrupt uplifting of a constant-\(Z\) surface. However, a circulation argument indicates that without a downdraft, this mechanism fails to produce a significant rotation about a vertical axis within a few hundred meters of the ground. To see this, consider a vertical rectangular circuit ABCD in the inflow and a horizontal rectangular circuit EFGH in the updraft formed by points on the same streamlines as shown in Fig. 2a. Because of symmetry and conservation of \(v\), the circulation around both circuits is the same. However, because the flow is rising, the horizontal circuit can only be at a greater height than the top of the vertical one. For example, let the environmental shear in the \(y\) direction be a constant \(2 \times 10^{-2} \text{s}^{-1}\), corresponding to a maximum vertical difference in \(y\) velocities of 10 m s\(^{-1}\) in the lowest 500 m of the inflow. Then in a purely rising flow, the difference in \(v\) in the updraft cannot exceed 10 m s\(^{-1}\) at the 500-m level, with proportionally lower limits at lower levels. In contrast, a downdraft, by transporting \(y\) momentum downward, can create large circulations around horizontal circuits that are very close to the ground (Fig. 2b).

3. Vorticity in steady two-dimensional, three-directional flow

We can make further insights when the flow is assumed to be steady. Let \(q\) be any positive-definite conserved variable. Since \(\mathbf{V} \cdot (\psi q \alpha) = 0\), we can introduce a streamfunction \(\psi\) defined by
When \( q = 1 \) (a valid choice), \( q \) disappears from the definition of \( \psi \). At the ground \( (z = 0) \) \( w = 0 \), so \( \psi \) is a constant there. Since \( \psi \) contains an arbitrary constant, we may set \( \psi = 0 \) at \( z = 0 \). For steady flow, the subscript \( 0 \) refers, not to initial values, but to the uniform conditions that exist along a streamline far upstream from the storm where the inflow is purely horizontal, and \( Z \) equates with \( z_0 \), the height of the streamline at upstream infinity. Furthermore, trajectories and streamlines coincide, and the advectons of conserved variables vanish, which implies

\[
\nu = \nu_0(\psi), \quad \theta = \theta_0(\psi), \quad q = q_0(\psi). \tag{16}
\]

Derivatives with respect to \( \psi \) and \( z_0 \) are linked by

\[
-\frac{d}{q_0\alpha_0 d\psi} = \frac{d}{u_0 dz_0}. \tag{17}
\]

Velocity and vorticity components are related by

\[
(\xi, \eta, \zeta) = \frac{1}{q_0\alpha_0} \frac{d\nu_0}{d\psi}(u, 0, w) + [0, -\nabla \cdot (q_0\alpha_0 \nabla \psi), 0], \tag{18}
\]

or

\[
(\xi, \eta) = \frac{d\nu_0}{d\psi}(u_0, 0, w) + [0, -\nabla \cdot (q_0\alpha_0 \nabla \psi), 0]. \tag{19}
\]

Note that the projections of the vortex lines and streamlines onto the \( x-z \) plane coincide. Furthermore, when the conserved quantity \( q_0^0 \cdot d\psi/d\psi \) is positive, the maximum values of \( \alpha \zeta \) and \( w \) on a streamline are collocated.

The momentum equation in the \( x-z \) plane is

\[
\nabla B = \mathbf{v} \times \mathbf{\omega} + c_p \pi \gamma \theta_0, \tag{20}
\]

where \( B \) is the Bernoulli function

\[
B = c_p \pi \theta_0 + g z + \frac{u^2 + v_0^2 + w^2}{2}. \tag{21}
\]

The dot product of (20) with \( \mathbf{v} \) shows that \( B \) is constant along a streamline in isentropic flow. Evaluation of the Bernoulli function far upstream gives

\[
B(\psi) = c_p \pi_0 + g z_0 + \frac{u_0^2 + v_0^2 + w_0^2}{2} = c_p \pi_0 + g z_0 + \frac{u_0^2 + v_0^2}{2}. \tag{22}
\]

The pressure variation along a streamline is therefore given by

\[
\nabla B = \mathbf{v} \times \mathbf{\omega} + c_p \pi \gamma \theta_0, \tag{20}
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where \( B \) is the Bernoulli function

\[
B = c_p \pi \theta_0 + g z + \frac{u^2 + v_0^2 + w^2}{2}. \tag{21}
\]
\[ \pi - \pi_0 = \frac{g(z_0 - z) + (u_0^2 - u^2 - w^2)/2}{c_p \theta_0}. \]  

(23)

Along a streamline, large (small) dynamic pressure is collocated with high (low) values of \((\xi^2 + \xi^2)\) as well as with strong (weak) winds because of the proportionality between \(u\) and \(\xi\) and between \(w\) and \(\xi\) in (18). Consequently, air parcels approaching a \("\text{stagnation high},\"\) such as the surface high at the leading edge of a density current, decelerate within a strong adverse pressure gradient and are compressed horizontally. Note that \"stagnation\" in this paper implies \(u = w = 0\), but not \(u = 0\). So, near the surface, the horizontal vorticity available for upward tilting by a gust front is greatly reduced before it is tilted. Consequently, uplifting of horizontal vortex lines by a density current does not lead to appreciable vertical vorticity just off the ground.

From (18), the Lamb vector is

\[ \mathbf{L} = \mathbf{\omega} \times \mathbf{v} = \left( q_0 \alpha \eta - \frac{1}{2} \frac{d v_0^2}{d \psi} \right) \mathbf{\nabla} \psi. \]  

(24)

Since \(B\) and \(\theta\) are functions of \(\psi\) alone, (20) reduces to

\[ q_0 \alpha \eta - \frac{1}{2} \frac{d v_0^2}{d \psi} = -\frac{d B}{d \psi} + c_p \frac{d \theta_0}{d \psi}. \]  

(25)

When \(q_0 = 1\) and \(v_0 = 0\), this equation is the dry version of one derived by Lilly [1979; see his (3.6)] while reviewing theoretical work on squall lines by Moncrieff and Green (1972) and Moncrieff (1978). According to (25), the difference \(d B\) in the Bernoulli function between close streamlines is equal to minus the Lamb vector \(\mathbf{L}\) plus \(T d S\), where \(d S\) is the difference in the streamlines’ entropies.

Subtraction of the upstream evaluation of (25) from (25) itself gives

\[ \eta = \frac{\alpha_0 \eta_0}{\alpha} + c_p (\pi - \pi_0) \frac{d \theta_0}{q_0 \alpha d \psi} = \frac{1}{2} \frac{d u_0^2}{q_0 \alpha d \psi} + c_p (\pi - \pi_0) \frac{d \theta_0}{q_0 \alpha d \psi}. \]  

(26)

Substituting for \(c_p (\pi - \pi_0)\) from (23) and using (17) then yields

\[ \eta = \frac{\alpha_0 \eta_0}{\alpha} + \frac{\alpha_0 g}{\alpha} \frac{d \ln \theta_0}{dz_0} \left( z - z_0 + \frac{u^2 + w^2 - u_0^2}{2 g u_0} \right). \]  

(27)

where the first and second terms on the rhs are the barotropic and baroclinic vorticity, respectively. [Incidentally, (27) can be derived from (12), as is done in the appendix.] Generally, wind speeds are moderate enough to satisfy \(|u^2 + w^2 - u_0^2|/2 \ll g|z - z_0|\) so that

\[ \eta = \frac{\alpha_0 \eta_0}{\alpha} + \frac{\alpha_0 g}{\alpha} \frac{d \ln \theta_0}{dz_0} (z - z_0), \]  

(28)

which is a special case of Moncrieff and Green’s (1972) vorticity equation [their Eq. (12)]. In the Boussinesq approximation \([\alpha = \alpha_0 = 1, q_0 = 1, \theta_0 = \text{constant} (= \theta_c)]\) except when multiplied by \(g\) that is valid for shallow flows, (27) reduces to (Davies-Jones 2006)

\[ \eta = -\nabla^2 \psi = \frac{\alpha_0 g}{\alpha \theta_c} \frac{d \theta_0}{dz_0} (z - z_0); \quad q_0 = 1. \]  

(29)

We can deduce the effects of environmental stratification from (28). The first term on the rhs is exactly the barotropic vorticity \(\eta_{BT}\) and the second term is approximately the baroclinic vorticity \(\eta_{BC}\). In a branch of flow where warm air enters horizontally from the east \((v_0 < 0)\) with positive shear \((\theta_0 > 0)\), rises, and exits to the east (so that \(z - z_0 \geq 0\), \(\eta_{BT}\) is positive. Because there is no stretching or tilting of \(\gamma\) vorticity, it changes only as a result of dilatation \(a/\alpha a_0\). In the same branch, \(\eta_{BC}\) has the opposite sign of \(\frac{d \theta_0}{dz_0}\)—the environmental stratification. We rule out the \(\frac{d \theta_0}{dz_0} < 0\) case on the grounds that the inflow would be unstable and break down into convective rolls. As in the simulations of Markowski et al. (2003), stable stratification should be less favorable for vertical-vorticity production than neutral stratification. This is evident from the factor \(- (z - z_0) \frac{d \theta_0}{dz_0}\) in the \(\eta_{BC}\) term. This factor is equal to the linearized restoring force when parcels are displaced vertically in a stably stratified environment (Dutton 1986, p. 71). The resistance to lifting weakens the circulation in the \(x-z\) plane and consequently decreases vertical vorticity because \(w\) is proportional to \(\xi\) in (18).

A linear elliptic partial differential equation for \(\psi\) can be obtained without approximating (27). Note that

\[ \eta = -\nabla \cdot (q_0 \alpha \nabla \psi) = -q_0 \nabla \cdot (\alpha \nabla \psi) - \alpha \nabla \psi \cdot \nabla q_0, \]  

(30)

where

\[ -\alpha \nabla \psi \cdot \nabla q_0 = -\alpha \frac{d q_0}{d \psi} \nabla \psi \cdot \nabla \psi = -\frac{u^2 + w^2}{2} \frac{d \ln q_0^2}{q_0 \alpha d \psi} = \frac{\alpha_0 g}{\alpha} \frac{u^2 + w^2}{2 u_0} \frac{d \ln q_0^2}{dz_0} \]  

(31)

via (17). With use of (30), (27) becomes
\[-q_0 \nabla \cdot (\alpha \nabla \psi) = \frac{\alpha_0 \eta_0 + \alpha g}{\alpha} \left[ \frac{g}{b_0} \frac{d \ln \theta_0}{dz_0} (z - z_0 - \frac{u_0^2}{2g}) + \frac{u^2 + w^2}{2u_0} \left( c_p \frac{d \ln \theta_0}{dz_0} - \frac{d \ln \theta_0^2}{dz_0} \right) \right]. \tag{32} \]

The choice \( q_0 = (c_p \theta_0)^{1/2} \) eliminates the nonlinear term (Lilly 1979, p. 150), leaving

\[-q_0 \nabla \cdot (\alpha \nabla \psi) = \frac{\alpha_0 \eta_0 + \alpha g}{\alpha} \frac{d \ln \theta_0}{dz_0} (z - z_0 - \frac{u_0}{2g}); \]
\[q_0 = (c_p \theta_0)^{1/2}. \tag{33} \]

This is an elliptic linear partial differential equation in \( \psi \) if specific volume \( \alpha \) is assumed to be a function of just \( z \) (which is essentially true) and inflow conditions \( u_0 \) and \( z_0 \) are specified. Reassuringly, the comprehensive (33) reduces to (29) in the Boussinesq approximation, so (29) is valid for shallow flows such as density currents.

The cold pool will influence the streamlines in the warm air, owing to the nature of solutions to the Poisson equation (29), whereby the streamfunction \( \psi(x, z) \) at a point is given by the weighted areal integral of the forcing function at all other points in the \( x-z \) plane. Here, the forcing function is the \( y \) vorticity and the weighting function is the Green’s function. The negative vorticity at the leading edge of the cold air certainly modifies the streamlines ahead of the gust front, as evident in RKW’s Fig. 20. This is illustrated in the next section by examination of a two-fluid flow.

4. Two-fluid model

RKW’s theory of deep lifting by gust fronts is derived from a steady-state vorticity-flux analysis of an inviscid two-fluid flow in a sheared, unstratified environment. The cold pool is stagnant in the reference frame moving eastward with its leading edge. RKW hypothesize an optimal state in which “the low-level easterly flow is turned by the cold pool in such a way as to exit as a vertically oriented jet” and “the import of the positive vorticity associated with the low-level shear just balances the net generation of negative vorticity by the cold pool in the volume” (p. 478). As already stated, their assumption that there is no vertical flux of vorticity through the top of their control volume seems problematic. Here, we examine their analysis in the light of the theory developed in sections 2 and 3.

The \( y \) vorticity in two-dimensional, inviscid, isentropic, Boussinesq, steady flow is given by (29). The barotropic vorticity is constant along streamlines. This is called vorticity conservation by Xu (1992) and Xu and Moncrieff (1994). Hence, in the vorticity budget, import of barotropic vorticity is balanced only by export of barotropic vorticity. In a two-fluid model, the temperature gradient is confined to the interface between the two fluids, which is also a vortex sheet (Xu 1992; Xu and Moncrieff 1994). Baroclinic vorticity is generated in the (infinite) horizontal temperature gradient at the leading (east) edge of the cool air. From (29) [or more generally from (12) for unsteady compressible flow], we see that the baroclinic vorticity is confined to the interface. RKW assume no vorticity flux through the west edge of the control volume, but the existence of the vortex sheet requires a westward baroclinic vorticity flux along the top of the interface. Therefore, in a steady state the baroclinic vorticity generated in the volume is exported from the west edge at its intersection with the interface. In steady, inviscid, isentropic flow, the vorticity budget envisioned by RKW seems impossible because the flow out of the top of the volume only transports barotropic vorticity, which is conserved along streamlines. Baroclinic vorticity can flow out through the top only if the head of the density current head extends above the top. The circulation associated with the vortex sheet acts to lift the lowest of the inflow streamlines over the head of the density current. Given sufficient shear and lift, other inflow streamlines may reverse (“U turn”) back over the inflow as they rise into the westerly current aloft.

The strength of the interfacial vortex sheet is determined as follows. The unit tangent and normal vectors in the \( x-y \) plane are \( \mathbf{s} \) and \( \mathbf{n} \), respectively, where \( \mathbf{s} \) is forward along the interface and \( \mathbf{n} \) is the upward normal 90° to the left of \( \mathbf{s} \). The strength of the vortex sheet (circulation per unit length) is

\[\gamma = \eta_{BC} \delta, \tag{34}\]

where \( \delta \) is the sheet thickness (as \( \delta \to 0, \eta_{BC} \to \infty \) so that the strength of the vortex sheet remains constant). For Boussinesq flow, the baroclinic vorticity is

\[\eta_{BC} = \frac{z - z_0}{u_0} \frac{db_0}{dz_0} \tag{35}\]

from (29), where the buoyancy variable is \( b = g\theta/\theta_c \). Here \( z_0 = 0 \) because the interfacial streamline \( \psi = 0 \) branches off the surface streamline at the stagnation point. Substituting \( -db_0/dz_0 \) for \( u_0 \) gives

\[\eta_{BC} = -\frac{z}{\Delta a} \frac{db_0}{d\psi}. \tag{36}\]

But \( db_0/d\psi = (db_0/d\psi)/(V \Delta \psi) \), where \( \Delta a = a_2 - a_1 \) is the jump in any variable \( a \) across the interface
from cool-side value \( a_1 \) to warm-side value \( a_2 \), and \( V = (V_1 + V_2)/2 \) is the wind speed averaged across the interface. Therefore, (34) becomes

\[
\gamma = -h_f(x) \frac{\Delta h_0}{V}. \tag{37}
\]

where \( h_f(x) \) is the height of the interface. Note that \( \gamma \) is independent of \( \delta \), so it is well defined in the limit \( \delta \to 0 \).

When the cold pool is stagnant, \( V_1 = 0 \) and \( V = V_2/2 = (h_1\Delta h_2/2)^{1/2} \), so \( \gamma = -(2h_1\Delta h_0)^{1/2} \).

The strength of a vortex sheet is equal to the jump in the \( s \) component of the wind (Batchelor 1967, p. 97) so \( \gamma \) is well defined in the limit \( \delta \to 0 \).

We now consider a semi-infinite vortex sheet with density-current head and a surface stagnation point at \((x, z) = (0, 0)\). We compute streamfunctions on a 251 \times 41\) grid over a window \([-8, 8] \times [0, 2]\) by approximating the source vortex sheet with hundreds of discrete source vortices of circulation \( \gamma \) \( ds \) where \( \gamma = -(2h_1)^{1/2} \) and \( ds \) is the vortex spacing \((0.01 \text{ to } 0.02) \). The source vortices are Rankine combined vortices with tiny core radii \((h_0 \text{ the grid spacing})\) to avoid singularities. We also include image potential vortices \((1 \text{ or } 400 \text{ per source vortex depending on the absence or presence of a lid}) \). We approximate the cold block as a semi-infinite rectangle of unit height or a streamlined shape \([h_f = 1 - \exp(\sqrt{3}x)]\). The interface of the latter has the correct 60° slope at the ground. To avoid “end effects” in the streamfunction plots, the vortices span the \( x \) interval \([-16, 0]\) for a semi-infinite density current or \([-16, 16]\) for an infinite vortex sheet \((x \text{ from } -\infty \text{ to } x = \infty) \). Our computations for the infinite vortex sheet with and without a lid confirm the theoretical predictions for these cases. When the ambient fluid is infinitely deep, both the vertical and streamlined fronts of the density currents are halted by imposing a uniform flow with \( U = -1.0 \). This agrees with the value predicted by Shin et al. (2004). However, the streamline that initially follows the interface from the stagnation point becomes higher than the interface near the top of the front and rearward from there (Figs. 3a,b). This and contours of horizontal velocity (not shown) indicate that cool fluid behind the front is traveling faster than the front; thus, the flow is unsteady, the cold pool is not stagnant, and its height is poised to increase.

With a lid at \( z = 2 \) and \( U = -0.707 \), the streamlined density current appears almost steady. The streamline through the surface stagnation point almost coincides with the interface and there is insignificant motion inside the cold block (Fig. 4a). Insertion of a lid at \( z = 2 \) slows down the front and the body of cold air to a common velocity. In the density current with the vertical front, the streamline is unable to conform closely to the interface and there is a closed streamline at the front with downward motion of 0.2 behind the front (Fig. 4b).
contrast, the minimum vertical velocity of the streamlined density current is just ~0.04. The upward motion in the warm air is also much greater for the vertical front. For both ambient fluid depths (2 and \( \infty \)), the maximum vertical velocity is about double that for the streamlined front (1.2 compared to 0.6). This suggests that models of deep lifting ahead of gust fronts should be initialized not with a vertical front but with a quasi-steady streamlined pseudocold front that makes a 60° angle with the ground.

5. Numerical simulation of tilting of strong environmental vorticity by a powerful density current

We ran a simple numerical simulation to see if the abrupt upward turning of horizontal vortex tubes at a gust front could produce significant vertical vorticity very close to the ground. The simulation is three dimensional, but there are no initial gradients in the \( y \) direction and none develop during the course of the
integration. In our attempt to make this happen, we chose an extreme case (or “worst-case scenario”) with a very strong cold pool and an environment with very large shear and no static stability. Except for the addition of vertical shear in the \(y\) direction, the simulation is similar to ones made by RKW (see their Figs. 19 and 20).

The dry version of the Bryan cloud model version 1 (CM1), release 16, is used (Bryan and Fritsch 2002). The model equations are discretized on a \(C\) grid (Arakawa and Lamb 1977) having dimensions of \(50 \times 10 \times 20\) km. The domain has rigid, free-slip top and bottom boundaries, open west and east boundaries, and periodic north and south boundaries. The horizontal and vertical grid spacing is 50 m. The advection scheme is fifth order, which has implicit diffusion. No additional artificial diffusion is included. Eddy viscosities are determined from the prognosed turbulent kinetic energy and a mixing length scale (Deardorff 1972). There are no surface fluxes, Coriolis force, or radiative transfer.

The simulation is initialized with a 5-km-deep block of cold air within the westernmost 10 km of the domain. The minimum potential temperature perturbation (found at the surface) within the cold-air block is \(-12\) K. The potential temperature perturbation decreases linearly with height within the cold air. The environment is otherwise neutrally stratified. The environmental vorticity available for tilting, \(\xi_0 = -dv_0/dz\), is \(-0.02\) s\(^{-1}\) (Fig. 5d). This corresponds to a southerly wind shear of \(20\) m s\(^{-1}\) \(\text{km}^{-1}\) (Fig. 5b). This shear is applied over the depth domain because \(v\) is a passive scalar and its contours serve as vortex lines for \((\xi, 0, \zeta)\). The component of environmental vorticity parallel to the gust front \(\eta_0\) is 0.02 s\(^{-1}\) in the lowest 1000 m (Fig. 5e). The northward vorticity component (i.e., westerly vertical shear; Fig. 5a) is included to offset the strong southward vorticity generated solenoidally by the density current and, hence, to maintain the density current’s almost vertical leading edge in accord with the RKW discovery. There is no zonal wind shear above 1 km.

**Fig. 5.** Vertical cross sections 10 min after a cold block is released into a neutrally stratified environment having strong westerly and southerly low-level shear. (a)–(g) Isopleths of potential temperature perturbation are shown every 2 K (starting at 1 K) in black, overlaid on (color shading) \(u, v, w, \xi, \eta, \zeta, \) and \(p'\), respectively. (h) Streamlines (black) are overlaid on the field of potential temperature perturbation (color shading). Axis labels indicate distances in kilometers.
There is a large rotor at the leading edge of the density current (Fig. 5h). This rotor, which also appears in the idealized simulations of RKW and WR, originates as a starting vortex created by the impulsive start to the simulation. The vortex is shed from the top edge of the vertical vortex sheet that forms immediately in response to the sudden introduction of the cold block (HärTEL et al. 1999). The starting vortex can cause large vertical displacements of parcels in the RKW and WR models. This may explain in part why the displacements appear larger at similar shear than those of Xu et al. (1996), whose model is initialized in a more balanced way by utilizing Xu’s (1992) inviscid steady-state solution. It should also be noted, however, that displacements in Xu et al. (1996) are restricted by the presence of the rigid lid, which progressively confines the flow with increasing shear.

After 10 min, the head of the density current is still over 3 km deep with its leading edge staying steep (Fig. 5). Just ahead of this almost vertical wall, warm air is rising rapidly with vertical velocities in excess of 20 m s\(^{-1}\) located as low as 1 km above ground (Fig. 5c). The peak vertical vorticity is 0.02 s\(^{-1}\) and is located well aloft at 3 km (Fig. 5f). Despite the large environmental horizontal vorticity in the lowest 1 km, the maximum vertical vorticity at 25 m (the lowest scalar level) in the warm air ahead of the density current is only 1.25 \(\times 10^{-3}\) s\(^{-1}\). The vertical vorticity is small there even though vortex lines are being tilted very abruptly near the surface by the nearly vertical density-current head (Fig. 5i). The reasons for this rather surprising result are contained in the pressure field (Fig. 5g) and in the velocity field (Fig. 5d). A stagnation high is present at the surface at the leading edge of the density current (Fig. 5g). Thus, warm parcels encounter an adverse pressure gradient and decelerate as they approach within about 2 km of the gust front. Consequently, they are compressed in the east–west direction (and stretched vertically to conserve mass). Owing to the east–west compression, the westward vorticities of these parcels are greatly reduced before the parcels encounter large gradients of vertical velocity. The magnitude of \(\xi\) in the lowest 100 m decreases from 0.02 to 0.002 s\(^{-1}\) (Fig. 5d) by the time the streamlines (Fig. 5h) turn upward at the density current’s leading edge. The vertical stretching of parcels implies that air rises gradually at first about 2 km ahead of the gust front. This is evident in the streamlines (Fig. 5h) and vortex lines (which coincide with \(v\) contours; Fig. 5b).

6. Summary

Based on the following line of reasoning, we conclude that tilting of horizontal vortex tubes by a gust front does not cause a tornado. For two-dimensional (\(\partial / \partial y = 0\), three-directional, inviscid, isentropic flow in a non-rotating atmosphere, the velocity component in the \(y\) direction, \(u\), is a conservative passive scalar. It serves as a “streamfunction” for the \((\xi, 0, \zeta)\) vector field so the vortex lines lie in surfaces of constant \(v\). If the environment is horizontally homogeneous and thus devoid of vertical vorticity and \(Z\) is the original height of a parcel, then the vortex lines lie in constant-\(Z\) surfaces, which coincide with the constant-\(v\) surfaces and the isentropic surfaces. Vorticity components \(\xi\) and \(\zeta\) are related by \(\zeta = \xi(\partial z / \partial x)_{Z,v, \text{or } \theta}\). This suggests that abrupt upturning of vortex lines by a density current or topography could produce appreciable vertical vorticity next to the ground. However, a circulation argument shows that an updraft by itself cannot produce at very low levels the large differences in horizontal velocity associated with significant rotation. Differential downward transport of \(y\) momentum (or angular momentum in an axisymmetric flow) is required.

If the flow is also steady, \(v\) and hence \(dv/d\psi\) are constant along a streamline. The \(x\) and \(z\) components of vorticity and wind satisfy a Beltrami relationship—namely, \(\alpha(\xi, 0, \zeta) = (u, 0, w)dv/d\psi\) so the streamlines coincide with the projections of the vortex lines onto the \(x\)–\(z\) plane. Vorticity components \(\xi\) and \(\zeta\) are related by \(\xi = (\partial z / \partial x)_{\psi}\). Along a streamline with \(dv/d\psi > 0\), the maximum values of \(\alpha \xi\) and \(w\) are collocated, and \((\xi, 0, \zeta)\) vanishes at a stagnation point. The Bernoulli function \(B\) is constant along a streamline, so

\[
B = c_p \theta_0 \sigma + gz + (u^2 + w^2)/2 = c_p \theta_0 \pi_0 + gz_0 + u_0^2 / 2.
\]

At the stagnation point at the front of the density current, the pressure is high. Along a streamline, large dynamic pressure is collocated with high values of \(\xi^2 + \zeta^2\). Air parcels approaching the stagnation high decelerate in strong adverse pressure gradient and are compressed horizontally. Along a streamline \(\alpha \xi\) is proportional to \(u\). So, near the surface, the horizontal vorticity available for upward tilting is greatly reduced before it is tilted. Consequently, uplifting of horizontal vortex lines by a density current does not lead to appreciable vertical vorticity just off the ground. A time-dependent numerical simulation verifies this finding and generalizes it to unsteady flow.

Therefore, one cannot argue that because there is large amount of horizontal vorticity in a surface-based layer in the environment, abrupt tilting of it at an “obstacle” (such as a gust front or topographical barrier) will produce similar strength vertical vorticity very close to the surface. Linear thinking (i.e., assuming that horizontal vorticity is unmodified from environmental values) is misleading in this case because the abrupt tilting is
unavoidably associated locally with a stagnation flow that greatly compresses the horizontal vortex tubes prior to tilting.

The extent of parcel lifting by the gust front is of course important in convection initiation and propagation. In idealized simulations, the maximum vertical displacement of parcels may be vitally sensitive to the way the model is initialized. Updrafts ahead of streamlined and fairly steady density currents are relatively small. If an unstreamlined block of cold air is introduced without approximately balancing the wind field to it, a starting vortex forms at the top of the initially vertical leading edge. The upward velocity and parcel lifting ahead of the gust front are increased by this vortex and decreased by the presence of a low artificial lid.

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APPENDIX

Derivation of (27) by Integrating the Vorticity Equation

From (10) the y vorticity equation in Lagrangian coordinates is

$$\frac{\partial (\alpha \eta)}{\partial \tau} = \alpha_0 \left[ \frac{\partial f (\theta_0), \zeta_p \theta_0}{\partial (z_0, X)} \right]$$

(A1)

for steady flow. The vorticity of a parcel does not change when it is far upstream so $f (\theta_0) = \pi_0$. Let \( \partial \Pi / \partial \tau = \pi - \pi_0 \). Then

$$\frac{\partial (\alpha \eta)}{\partial \tau} = \left[ \frac{\partial (\Pi, \zeta_p \theta_0)}{\partial (z_0, X)} \right]$$

(A2)

because $\theta_0$ is conserved. Integration of (A2) yields

$$\frac{\alpha \eta - \eta_0}{\alpha_0} = \frac{\partial (\Pi, \zeta_p \theta_0)}{\partial (z_0, X)} = \frac{\partial (\Pi, \zeta_p \theta_0)}{\partial z_0} = \frac{-c_p \theta_0}{g} N_0^2 \frac{\partial \Pi}{\partial X}.$$  

(A3)

But in steady flow,

$$\pi - \pi_0 = \frac{\partial \Pi}{\partial t} = q_0 \alpha \frac{\partial (\Pi, \psi)}{\partial (z, x)} = q_0 \alpha_0 \frac{\partial (\Pi, \psi)}{\partial (z_0, X)}$$

$$= -q_0 \alpha_0 \frac{\partial \Pi}{\partial z_0} \frac{\partial X}{\partial \eta} = u_0 \frac{\partial \Pi}{\partial X}.$$  

(A4)

Therefore, (12) becomes

$$\frac{\alpha \eta - \eta_0}{\alpha_0} = -\frac{c_p \theta_0 (\pi - \pi_0) N_0^2}{g}$$

(A5)

After substituting for $\pi - \pi_0$ from (23), we obtain (27):

$$\eta = \frac{\alpha_0}{\alpha} \eta_0 + \frac{\alpha_0}{\alpha} \left( z - z_0 + \frac{u^2 + w^2 - u_0^2}{2g} \right) N_0^2.$$  

(A6)

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