Dr. Droegemeier

METR 3113 – Atmospheric Dynamics I
Fall 2016

Problem Set #3

Distributed Wednesday, 21 September 2016
Due Friday, 30 September 2016

INSTRUCTIONS: For each of the problems below, apply all 6 steps in the problem-solving handout. Pay close attention to neatness, and describe your work at each step of the solution process.

1. The Divergence Operator. Assuming 3-D Cartesian coordinates, show that the mass continuity equation, in so-called mass divergence form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = 0$$

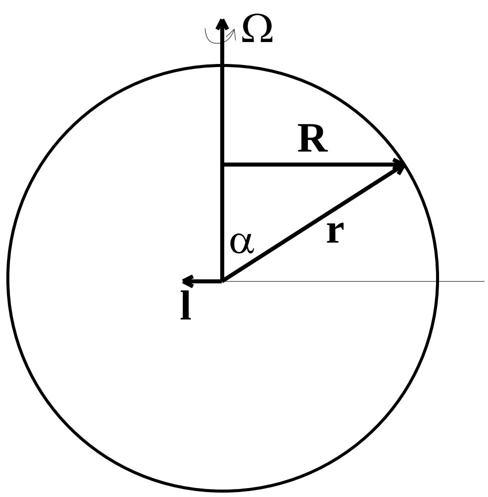
also may be written in so-called velocity divergence form as

$$\frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{U} = 0$$

2. Earth's Rotation. Assuming $\mathbf{r}$ to be a position vector of a point on the Earth's surface in a geocentric system and $\mathbf{\Omega}$ to be the vector of the Earth's angular velocity, show that:

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\Omega^2 \mathbf{R} ,$$

where $\mathbf{R}$ is a vector that is perpendicular to the axis of Earth's rotation and has a magnitude equal to the distance to the axis of rotation (see the plot below). <u>Give a physical interpretation of the result.</u>



3. Divergence and Curl Operators. Assuming 3-D Cartesian coordinates, show that

$$(\mathbf{U}\cdot\nabla)\mathbf{U} = \frac{1}{2}\nabla(\mathbf{U}\cdot\mathbf{U}) - \mathbf{U}\times(\nabla\times\mathbf{U})$$

where $\mathbf{U}$ is the three-dimensional wind vector.

4. Vector Identities. Confirm the following identities for 3-D Cartesian vectors by considering them in coordinate form, i.e., as $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, and using rules of vector calculus.

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$,

$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$,

$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$,

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = -\vec{c}\cdot(\vec{b}\times\vec{a}) = -\vec{b}\cdot(\vec{a}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$,

$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$,

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{b}\cdot\vec{d})(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$,

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{c}(\vec{a}\cdot\vec{b}\times\vec{d}) - \vec{d}(\vec{a}\cdot\vec{b}\times\vec{c})$.

5. Vectors in Rotated Coordinates. In 3-D Cartesian coordinates (X, Y, Z) vector $\vec{a}$ is given by $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and vector $\vec{b}$ is given by $\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$.

a. Which value does the dot product of these two vectors have in (i) the original coordinates and in (ii) the coordinates rotated with respect to the original coordinates by the 53.37° angle around the z axis?

b. Find (i) components and (ii) magnitude of the cross product $\vec{c} = \vec{a}\times\vec{b}$ of these two vectors in the original coordinates.

6. Gradient Operator and Coordinate Rotation. A frontal line on a plane is directed from SW to NE. A spatially-uniform horizontal temperature gradient on the same plane has the magnitude of 10^{-4} K m^{-1} and is directed NW. Find the rate of temperature change along the frontal line.

7. Vector Operators. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. Find the value of the operation below and indicate whether it is a scalar or a vector.

$\nabla\cdot\mathbf{r} =$

$\nabla\times\mathbf{r} =$

$\nabla^2 r^2 =$