Lecture 9. September 12, 2016

Topics: Scalars and vectors. Notation. Vector addition. Vector subtraction. Unit vector. Scalar (dot) product. Unit vectors in Cartesian coordinates. Vector magnitude. Vector projection on a coordinate axis (vector component). Multiplication of vector by a scalar. Examples of operations with vectors.

Reading: Section 1.1 of Holton and Hakim, Section 3 of Fiedler.

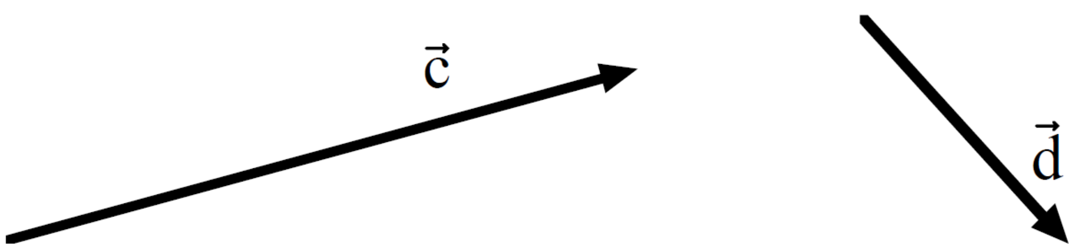
1. Scalars and vectors

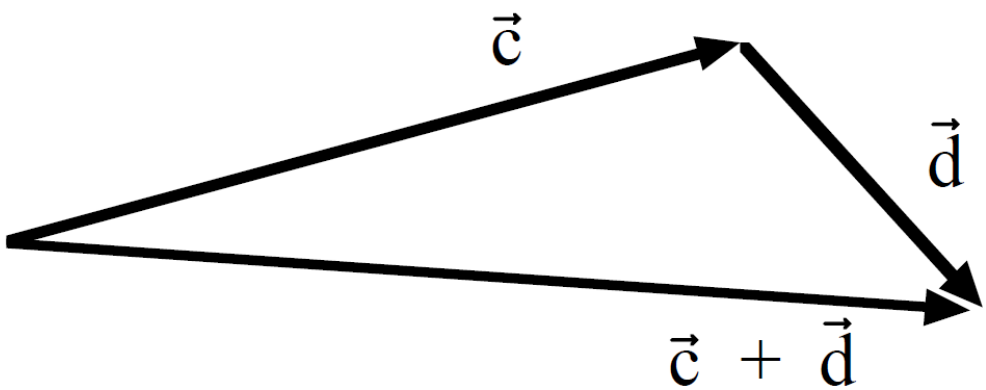
A physical quantity (or variable) that has magnitude only is called a ***scalar*** (see Class 2). Examples: mass m, temperature T, pressure p, density ρ, wind speed V, gas constant of the dry air R_d. Scalars can be constant or functions of space and time.

A physical quantity (or variable) that has magnitude and direction is called a ***vector*** (see Class 2). It can be represented in space (or on a plane) by an arrow. An arrow (over-arrow to be more specific) is often used also to denote a vector quantity as $\vec{b}$ or $\vec{\Omega}$. In this class, we will be denoting vectors either in this manner or using a bold symbol, i.e., like $\mathbf{b}$ or $\mathbf{\Omega}$. Examples of vector quantities: acceleration $\vec{a}$ (or **a**), velocity $\vec{u}$ (or **u**), gradient of a scalar field, e.g., of temperature, ∇T (the meaning of the gradient as a vector operation will be discussed during next classes), Earth's angular velocity $\vec{\Omega}$ (or $\mathbf{\Omega}$).

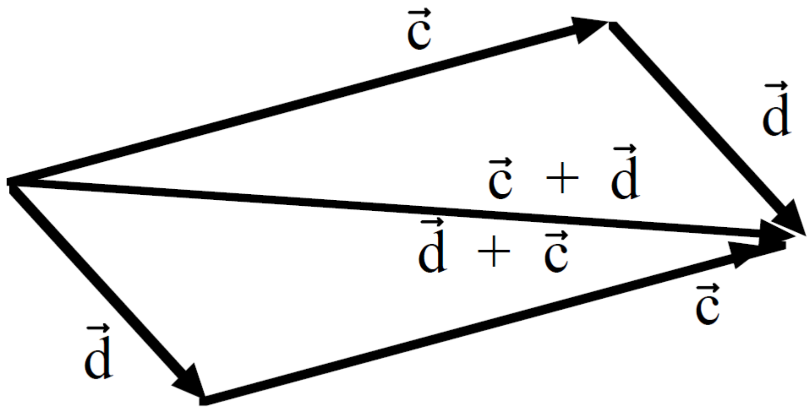
2. Vector addition

Consider two vectors, $\vec{c}$ and $\vec{d}$:

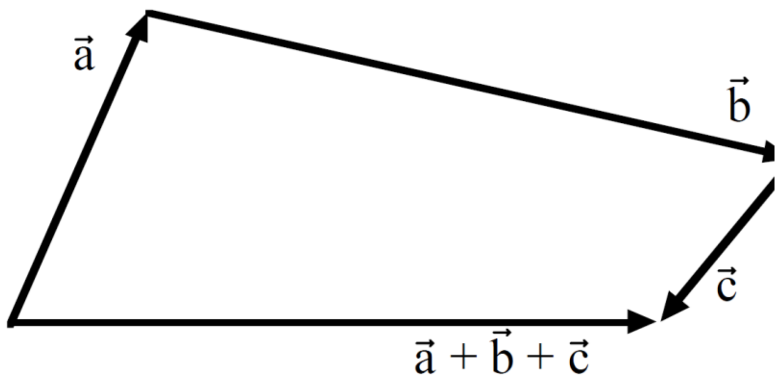


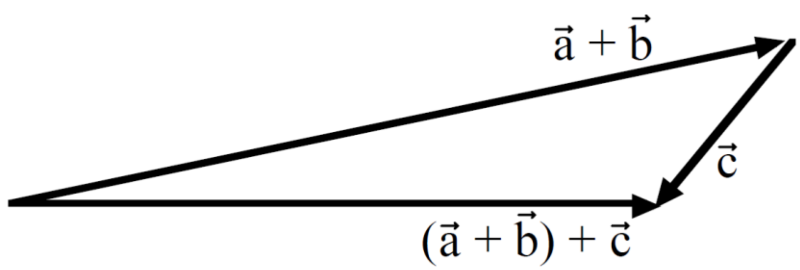


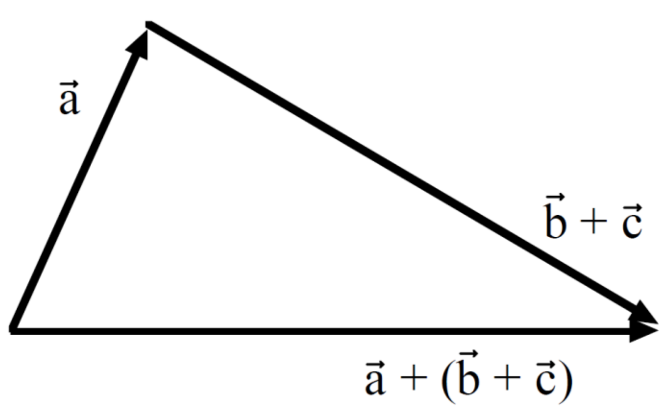
To add $\vec{d}$ to $\vec{c}$, place the tail of the second vector to the tip of the first. The sum of two vectors, vector $\vec{c}+\vec{d}$, is the vector that extends from the tail of $\vec{c}$ to the tip of $\vec{d}$. The same vector results from by adding $\vec{c}$ to $\vec{d}$, so $\vec{c}+\vec{d}=\vec{d}+\vec{c}$, which indicates that vector addition is *commutative* (see illustration below).



Vector addition is also *associative*. This property is illustrated below, where $\vec{a}+\vec{b}+\vec{c}=(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.



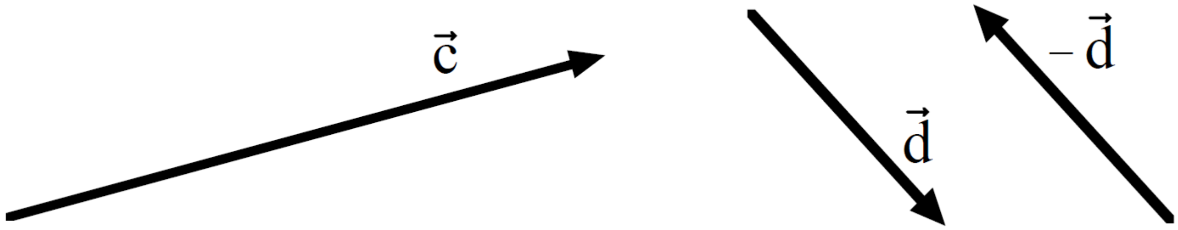


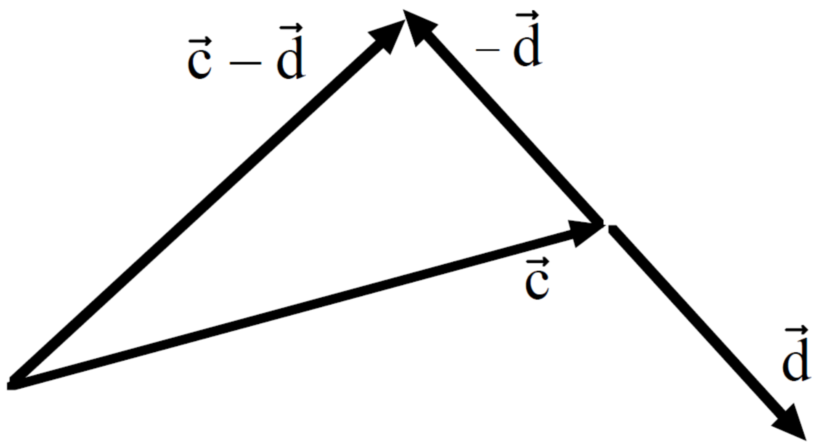


3. Vector subtraction

This operation is closely related to the vector addition: to subtract vector $\vec{d}$ from vector $\vec{c}$ add negative of vector to vector $\vec{c}$ (see illustration below), that is

$$\vec{c} - \vec{d} = \vec{c} + (-\vec{d}) .$$





4. Unit vector

A unit vector is defined as a dimensionless vector that has a magnitude of 1. The *magnitude* of the vector $\vec{b}$ (in Cartesian coordinates it would be a measure of its length, see p. **6**) is written as $b = |\vec{b}|$. The *unit vector* $\hat{b}$ (**note** the hat) in the direction of vector $\vec{b}$ is obtained through dividing $\vec{b}$ by its magnitude:

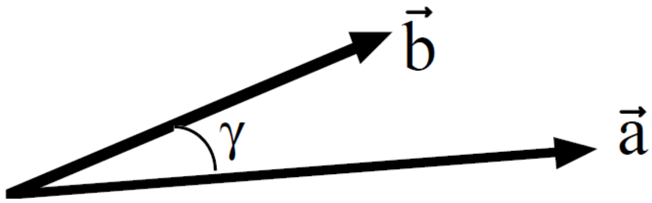
$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{b}}{b},$$

which also means that vector $\vec{b}$ may be written as a product of its magnitude and associated unit vector:

$$\vec{b} = b\hat{b} = |\vec{b}|\hat{b} .$$

5. Scalar/dot product of two vectors

Consider two vectors, $\vec{a}$ and $\vec{b}$ with angle γ between them (chosen as the smallest of the two angles).



The scalar (dot) product of $\vec{a}$ and $\vec{b}$ is defined as the following *scalar* quantity:

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\gamma=ab\cos\gamma\,.$$

The scalar product is *commutative*:

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}=|\vec{b}||\vec{a}|\cos\gamma=ba\cos\gamma\,,$$

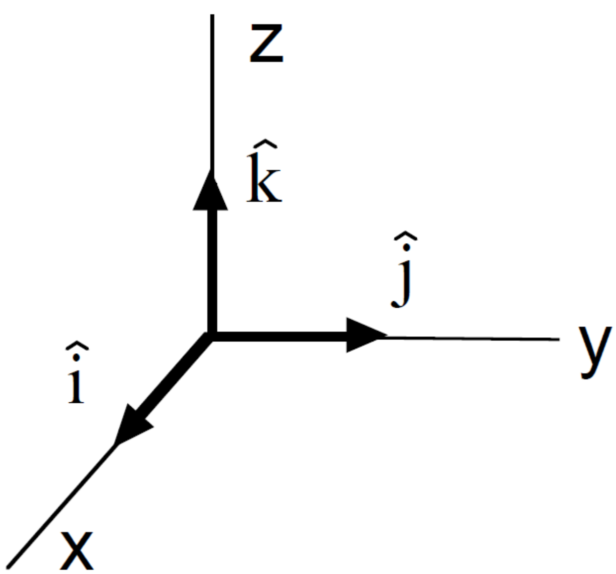
and *distributive*:

$$\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}\,,$$

From the definition of the scalar product it follows that for parallel vectors $\vec{a}$ and $\vec{b}$ (with $\gamma=0$) its value is just $|\vec{a}||\vec{b}|=ab$, and that the scalar product of a vector by itself, $\vec{b}\cdot\vec{b}=|\vec{b}|^2=b^2$ is equal to the square of the vector magnitude, the positively defined quantity.

Zeroness of the dot product, $\vec{a}\cdot\vec{b}=0$ may be a result of $\vec{a}=0$, $\vec{b}=0$ (or both vectors being zero vectors), or vectors being perpendicular to each other (in the latter case $\cos\gamma=0$).

6. Vectors in the Cartesian coordinate system



Consider a right-handed Cartesian coordinate system (*X*, *Y*, *Z*) with unit vectors $\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}}$ (constituting the so-called orthogonal basis) directed along the respective coordinates (see illustration above). From correspondence of the unit vector directions to the Cartesian coordinate directions, it follows that vectors $\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}}$ are perpendicular to each other, i. e., in terms of the scalar product:

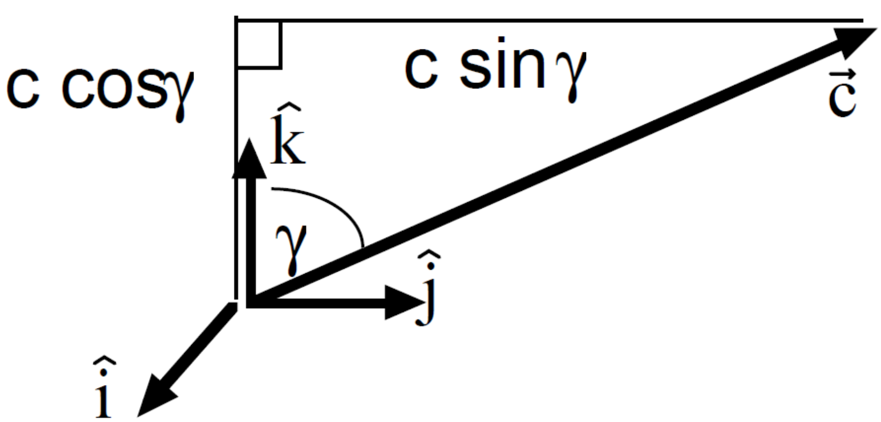
$$\hat{\mathrm{i}} \cdot \hat{\mathrm{i}} = 1,\ \hat{\mathrm{i}} \cdot \hat{\mathrm{j}} = 0,\ \hat{\mathrm{i}} \cdot \hat{\mathrm{k}} = 0,$$

$$\hat{\mathrm{j}} \cdot \hat{\mathrm{j}} = 1,\ \hat{\mathrm{j}} \cdot \hat{\mathrm{i}} = 0,\ \hat{\mathrm{j}} \cdot \hat{\mathrm{k}} = 0,$$

$$\hat{\mathrm{k}} \cdot \hat{\mathrm{k}} = 1,\ \hat{\mathrm{k}} \cdot \hat{\mathrm{i}} = 0,\ \hat{\mathrm{k}} \cdot \hat{\mathrm{j}} = 0..$$

The *projection* of a vector $\vec{\mathrm{c}}$ on a particular coordinate direction (also called the *component* of a vector in this direction) is defined as the dot (scalar) product of $\vec{\mathrm{c}}$ with the unite vector in that particular direction. The projection/component is therefore a scalar. In 3-D Cartesian coordinate system, vector $\vec{\mathrm{c}}$ has therefore the following components:

$$c_x = \vec{\mathrm{c}} \cdot \hat{\mathrm{i}},\ c_y = \vec{\mathrm{c}} \cdot \hat{\mathrm{j}},\ c_z = \vec{\mathrm{c}} \cdot \hat{\mathrm{k}}.$$



The vector projection may also be written (and interpreted) as the product of the vector magnitude and the cosine of the angle between the vector and the corresponding coordinate axis. For instance (see the plot above),

$$c_z = \vec{\mathrm{c}} \cdot \hat{\mathrm{k}} = c\left|\hat{\mathrm{k}}\right|\cos\gamma = c\cos\gamma\ .$$

The vector quantities $c_x\hat{\mathrm{i}},\ c_y\hat{\mathrm{j}},\ c_z\hat{\mathrm{k}}$, which may be interpreted as vector constituents of $\vec{\mathrm{c}}$, sum up, using the rules of the vector addition, into the vector $\vec{\mathrm{c}}$ itself:

$$\vec{\mathrm{c}} = c_x\hat{\mathrm{i}} + c_y\hat{\mathrm{j}} + c_z\hat{\mathrm{k}}.$$

The *magnitude* $b = \left|\vec{\mathrm{b}}\right|$ of vector $\vec{\mathrm{b}}$ in Cartesian coordinates is given by

$$b = \left|\vec{\mathrm{b}}\right| = \sqrt{b_x^{\ 2} + b_y^{\ 2} + b_z^{\ 2}}\ .$$

Another common way to present Cartesian vector $\vec{\mathrm{b}} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}$ is to write it as $\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$.

Vector $\vec{\mathrm{c}} = -\vec{\mathrm{b}}$, negative of vector $\vec{\mathrm{b}} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}$, in the coordinate form will appear as

$$c_x\hat{\mathrm{i}} + c_y\hat{\mathrm{j}} + c_z\hat{\mathrm{k}} = \vec{\mathrm{c}} = -b_x\hat{\mathrm{i}} - b_y\hat{\mathrm{j}} - b_z\hat{\mathrm{k}},$$

or

$$\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} -b_x \\ -b_y \\ -b_z \end{pmatrix},$$

so that $\vec{\mathrm{c}} + \vec{\mathrm{b}} = 0$.

7. Examples of operations with Cartesian vectors

Let $\vec{\mathrm{b}}$ be a Cartesian vector and *a* be a scalar. Then, $a\vec{\mathrm{b}} = ab_x\hat{\mathrm{i}} + ab_y\hat{\mathrm{j}} + ab_z\hat{\mathrm{k}}$.

Consider Cartesian vectors $\vec{\mathrm{a}} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}$, $\vec{\mathrm{b}} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}$, and their sum $\vec{\mathrm{c}} = \vec{\mathrm{a}} + \vec{\mathrm{b}} = c_x\hat{\mathrm{i}} + c_y\hat{\mathrm{j}} + c_z\hat{\mathrm{k}}$.

The sum of $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$ in component form may be written as

$$\vec{\mathrm{a}} + \vec{\mathrm{b}} = (a_x + b_x)\hat{\mathrm{i}} + (a_y + b_y)\hat{\mathrm{j}} + (a_z + b_z)\hat{\mathrm{k}},$$

so components of $\vec{\mathrm{c}} = \vec{\mathrm{a}} + \vec{\mathrm{b}}$ are given by

$$c_x = a_x + b_x,\ c_y = a_y + b_y,\ c_z = a_z + b_z.$$

Scalar (dot) product $\vec{\mathrm{a}} \cdot \vec{\mathrm{b}}$ of two Cartesian vectors $\vec{\mathrm{a}} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}$ and $\vec{\mathrm{b}} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}$ is given by

$$\vec{\mathrm{a}} \cdot \vec{\mathrm{b}} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}).$$

Using the dot product properties (p. **5**), we have

$$\vec{\mathrm{a}} \cdot \vec{\mathrm{b}} = a_x\hat{\mathrm{i}} \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}) + a_y\hat{\mathrm{j}} \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}) + a_z\hat{\mathrm{k}} \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}})$$

$$= a_xb_x\hat{\mathrm{i}} \cdot \hat{\mathrm{i}} + a_xb_y\hat{\mathrm{i}} \cdot \hat{\mathrm{j}} + a_xb_y\hat{\mathrm{i}} \cdot \hat{\mathrm{k}} + a_yb_x\hat{\mathrm{j}} \cdot \hat{\mathrm{i}} + a_yb_y\hat{\mathrm{j}} \cdot \hat{\mathrm{j}} + a_yb_z\hat{\mathrm{j}} \cdot \hat{\mathrm{k}} + a_zb_x\hat{\mathrm{k}} \cdot \hat{\mathrm{i}} + a_zb_y\hat{\mathrm{k}} \cdot \hat{\mathrm{j}} + a_zb_y\hat{\mathrm{k}} \cdot \hat{\mathrm{k}},$$

which provides

$$\vec{\mathrm{a}} \cdot \vec{\mathrm{b}} = a_xb_x + a_yb_y + a_zb_z.$$

If $\vec{a} = \vec{b}$, the above expression produces already considered (in p. **5**) formula for the vector magnitude. Indeed, in this case:

$$\vec{b} \cdot \vec{b} = b^2 = b_x{}^2 + b_y{}^2 + b_z{}^2 \text{ and } b = |\vec{b}| = \sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2} \ .$$

Note that for *perpendicular* vectors $\vec{a}$ and $\vec{b}$ ($\cos\gamma = 0$; see p. **5**):

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = 0 \ ,$$

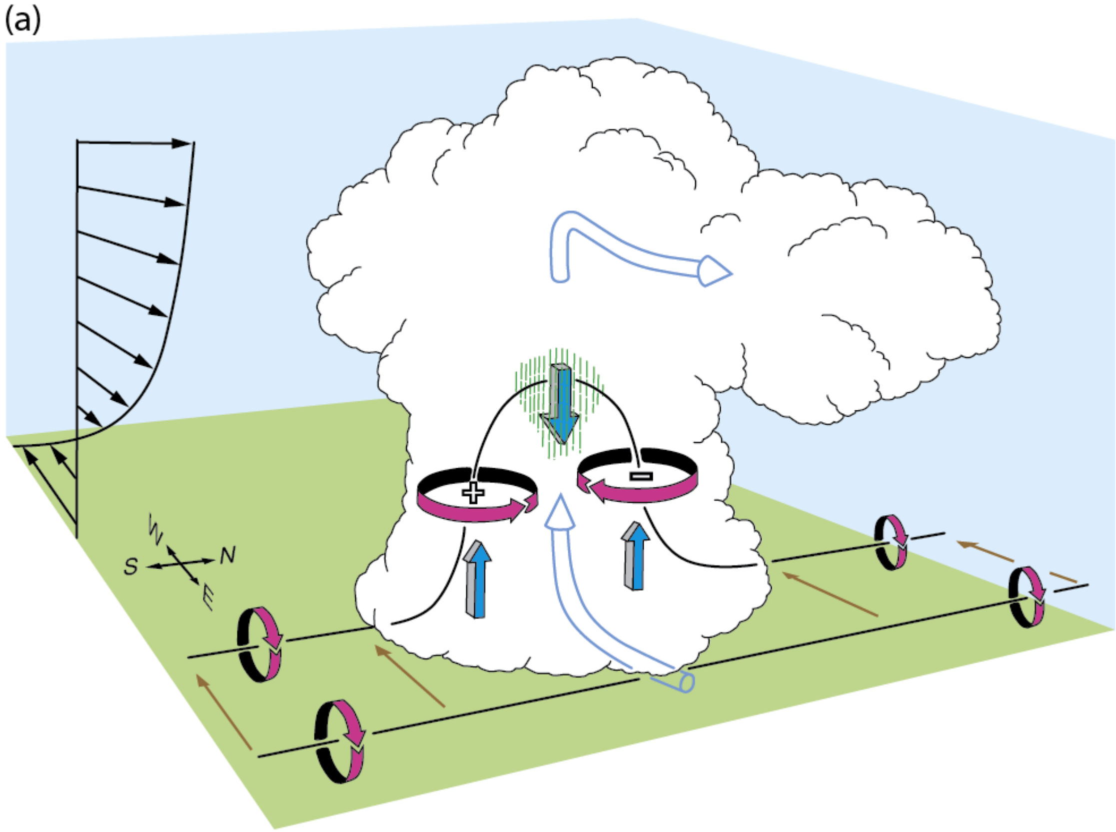
and for *parallel* vectors $\vec{a}$ and $\vec{b}$ ($\cos\gamma = \pm 1$; see p. **5**):

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \pm ab \ ,$$

where the plus sign corresponds to the vectors pointing in the same direction. Using $a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$ and $b = \sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}$, the condition of the vector parallelism becomes

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z} \ ,$$

(see also Class 6), with all ratios in the above expression being positive for vectors pointing in the same direction and negative for vectors pointing in the opposite directions.



Streamwise Vorticity

$$\vec{\omega}_s = \frac{(\vec{V} - \vec{C}) \cdot \nabla \times \vec{V}}{|\vec{V} - \vec{C}|} = \frac{(\vec{V} - \vec{C}) \cdot \vec{\omega}}{|\vec{V} - \vec{C}|}$$

Relative Helicity

$$(\vec{V} - \vec{C}) \cdot \vec{\omega} = |\vec{V} - \vec{C}||\vec{\omega}|\cos\theta$$

$$RH = \frac{(\vec{V} - \vec{C}) \cdot \vec{\omega}}{|\vec{V} - \vec{C}||\vec{\omega}|}$$

$$RH = \cos\theta$$