Lecture 7. September 7, 2016

Topics: Representation of wind components in Cartesian coordinate systems rotated around the vertical axis. Rotation of Ekman-model wind-component profiles. Wind hodograph.

Reading: Section 8.3.4 of Holton and Hakim.

1. Ekman model of wind profile in the lower atmosphere

In 1905, a Swedish/Norwegian oceanographer Vagn Walfrid Ekman suggested the following two-equation idealized model to quantitatively describe vertical distribution of mean (understood here as averaged over time) wind in the lower portion of the Earth’s atmosphere (this portion is commonly referred to as the atmospheric planetary boundary layer):

$$k\frac{\partial^2 u}{\partial z^2} + f(v - V_g) = 0,$$

$$k\frac{\partial^2 v}{\partial z^2} - f(u - U_g) = 0.$$

In these equations, the wind motion is specified in a local 3-D Cartesian coordinate system, where

- z is the vertical Cartesian coordinate directed against the vector of gravitational acceleration,
- x is the first horizontal Cartesian coordinate (typically it is directed east in meteorological applications),
- y is the second horizontal Cartesian coordinate (typically it is directed north in meteorological applications),
- k = const is the so-called eddy viscosity (or eddy diffusivity for momentum) that represents the degree of vertical exchange/mixing of momentum in the atmosphere,
- f = const is the so-called Coriolis parameter, (related to the magnitude of the Earth’s rotation vector Ω and geographic latitude φ as $f = 2\Omega\sin\varphi$), which accounts for the effect of the Earth’s rotation on the atmospheric motions (terms fv and $-fu$ are components of the Coriolis acceleration; this acceleration will be extensively discussed in the subsequent sections of the course),
- u is the mean flow (wind) component along x (it is thus the rate of x change in time, $u = \frac{dx}{dt}$),
- v is the mean flow (wind) component along y (therefore, $v = \frac{dy}{dt}$) and
- U_g, V_g, respectively, are the x and y components of the time- and height-constant geostrophic wind which results from the balance between the pressure gradient force and the Coriolis force.

According to the Ekman model, u tends to U_g and v tends to V_g at large z (i.e., the wind tends to be geostrophic with increasing distance from the ground), while u = 0 and v = 0 at z = 0 (i.e., there is no air motion directly at the ground).

The Coriolis parameter f is positive in the northern hemisphere and negative in the southern hemisphere. For the sake of certainty, we will place our consideration and coordinate system in the northern hemisphere.

2. Solutions of the Ekman model equations

Introducing a (constant) parameter $a_e = \sqrt{\dfrac{f}{2k}}$, the Ekman model equations (which are linear second-order ordinary differential equations) may be solved with respect to u and v to yield the following profiles of the wind components:

$$u(z) = U_g - e^{-a_e z}(U_g \cos a_e z + V_g \sin a_e z)\,,$$

$$v(z) = V_g - e^{-a_e z}(V_g \cos a_e z - U_g \sin a_e z)\,.$$

One may notice that parameter a_e has dimension of the inverse length. It is known to have a meaning of the inverse boundary-layer depth scale as we will see later.

3. Converting Ekman wind components to a rotated Cartesian system

Imagine that we need to convert components of wind, given by the Ekman model, to a Cartesian system rotated in the (X, Y) plane by angle α with respect to the original system. In other words, we would need to calculate wind components $u' = \dfrac{dx'}{dt}$ and $v' = \dfrac{dy'}{dt}$ in the new Cartesian system (X', Y', Z).

To rotate the coordinates, one can use the following formulas that we obtained earlier (Class 4):

$$x' = x\cos\alpha + y\sin\alpha\,,$$

$$y' = y\cos\alpha - x\sin\alpha\,,$$

which, after differentiating over t, yield the rotated velocity components

$$u' = u\cos\alpha + v\sin\alpha\,,$$

$$v' = v\cos\alpha - u\sin\alpha\,.$$

Suppose now that we would like to choose the new coordinate directions in such a way that the X' axis is directed along the geostrophic wind. This means turning the coordinates by the angle $\alpha = \arctan(V_g / U_g)$. The converting expressions

$$U_g' = U_g \cos\alpha + V_g \sin\alpha\,,$$

$$V_g' = V_g \cos\alpha - U_g \sin\alpha\,,$$

provide in this case:

$$G \equiv U_g' = U_g / \cos\alpha, \; V_g' = 0,$$

so geostrophic wind in (X', Y', Z) has only the x' component.

To obtain the above expressions for geostrophic wind components in the rotated coordinates, write $\tan\alpha = V_g / U_g$ and substitute $V_g = U_g \tan\alpha = U_g \dfrac{\sin\alpha}{\cos\alpha}$ into expressions for U_g' and V_g'.

It follows from

$$u(z) = U_g - e^{-a_e z}(U_g \cos a_e z + V_g \sin a_e z), \qquad v(z) = V_g - e^{-a_e z}(V_g \cos a_e z - U_g \sin a_e z)$$

that in the new coordinate system, where X' is aligned with the geostrophic wind, $U_g' = G$, and $V_g' = 0$, the wind components will appear as

$$u'(z) = G(1 - e^{-a_e z} \cos a_e z), \; v'(z) = G e^{-a_e z} \sin a_e z.$$

Note that the magnitude of the geostrophic wind (that is commonly called the geostrophic wind speed) is the same in both coordinate systems:

$$\sqrt{U_g^{\,2} + V_g^{\,2}} = G = \sqrt{U_g'^2 + V_g'^2}.$$

4. Wind distribution in the atmosphere according to the Ekman model

- At z=0, both wind components are 0, in accordance with the lower boundary conditions applied. They both grow with height at different rates as long as $a_e z$ is smaller that $\pi/4$. At this elevation: $u' = G(1 - e^{-\pi/4}/\sqrt{2})$ and $v' = G e^{-\pi/4}/\sqrt{2}$.
- When $a_e z$ reaches $\pi/2$, we have $u' = G$ and $v' = G e^{-\pi/2}$, which means that wind at this level is super-geostrophic (its magnitude/speed is larger than that of the geostrophic wind).
- At $a_e z = \pi$: $u' = G(1 + e^{-\pi})$ and $v' = 0$, which means that at this elevation the wind direction coincides with the geostrophic wind direction. Due to this, level $z = \pi / a_e = \pi\sqrt{\dfrac{2k}{f}}$ is often taken as the *planetary boundary layer top* within the Ekman model framework. The wind magnitude at $a_e z = \pi$ remains super-geostrophic.
- When $a_e z = 3\pi/2$: $u' = G$, $v' = -G e^{-3\pi/2}$, and the wind is still super-geostrophic.
- With $a_e z = 2\pi$: $u' = G(1 - e^{-2\pi})$ and $v' = 0$, and the wind is sub-geostrophic.

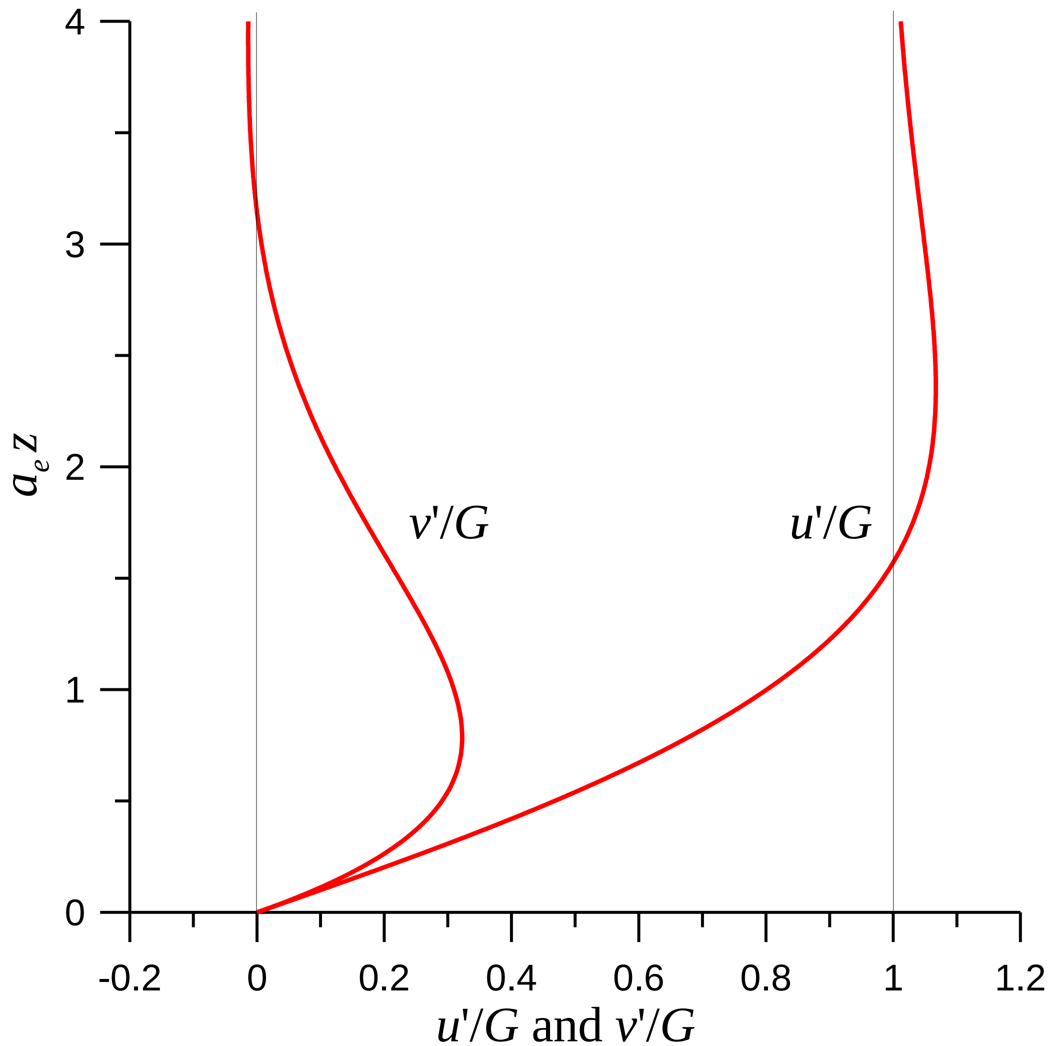
5. Wind hodograph in the Ekman model

A line diagram (on a plane) that connects tips of the wind velocity vector at different heights (or at different time moments) is called the *wind hodograph*. The hodograph of the wind vector as represented by the Ekman model is called the *Ekman spiral* (it is clear from the attached plot below why it is called the spiral).

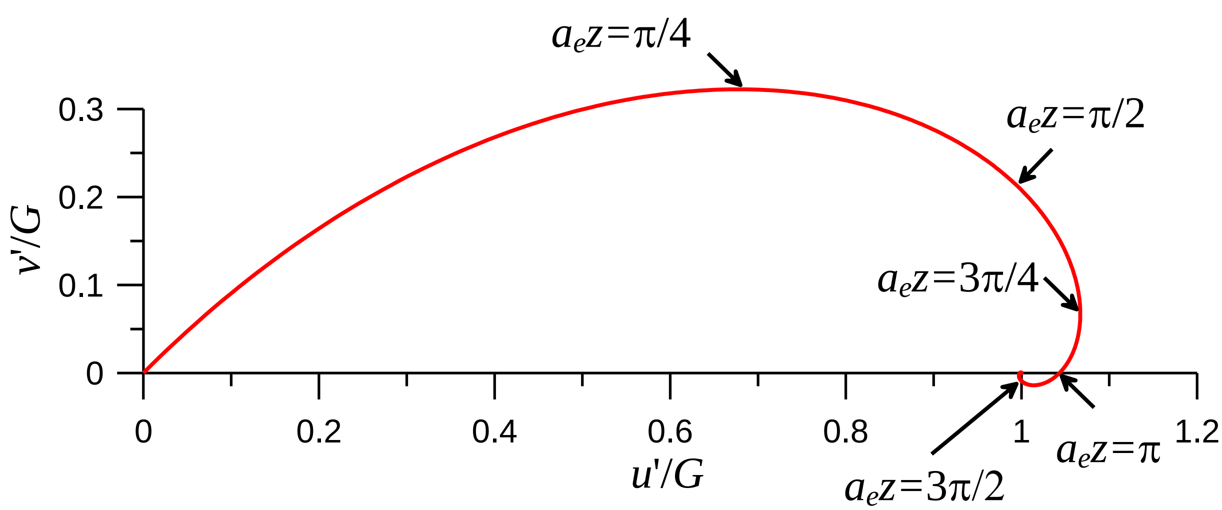
The *wind turn angle* α_w in the atmosphere is the angle between direction of geostrophic wind and the wind direction at given height. In the coordinate system with *X* axis directed along the geostrophic wind, this would be the angle between the *X* axis and the wind vector. The tangent of α_w in this case is given by

$$\tan\alpha_w = v'/u' = e^{-a_e z}\sin a_e z/(1-e^{-a_e z}\cos a_e z) .$$

Using the l'Hôpital's rule, one can demonstrate that in the very close vicinity of the surface, i.e., at *z* tending to zero, $\tan\alpha_w$ tends to 1. This means that near-surface wind in the Ekman model is turned left (counterclockwise when looking down *Z*) by $\pi/4 = 45^{\circ}$ angle with respect to the geostrophic wind, and $v' = u'$.



Wind-component profiles in (X', Y', Z) system according to the Ekman model



Wind hodograph in (X', Y', Z) system according to the Ekman model