Lecture 6. September 2, 2016

Topics: Operations with coordinate systems. Cartesian coordinate rotation. Vector in rotated Cartesian plane coordinates. Angle between two straight lines in 2-D Cartesian system and angle between two vectors. Circle in 2-D Cartesian and polar coordinate systems.

Reading: Sections 1.1 to 1.3 of Holton and Hakim. Sections 3 and 10 of Fiedler.

1. Cartesian coordinate rotation

Demonstrate that rotating 2-D Cartesian coordinate system by an angle α counterclockwise transforms point (x, y) into (x', y') with the old and new coordinates related as

$$x' = x\cos\alpha + y\sin\alpha,$$

$$y' = y\cos\alpha - x\sin\alpha,$$

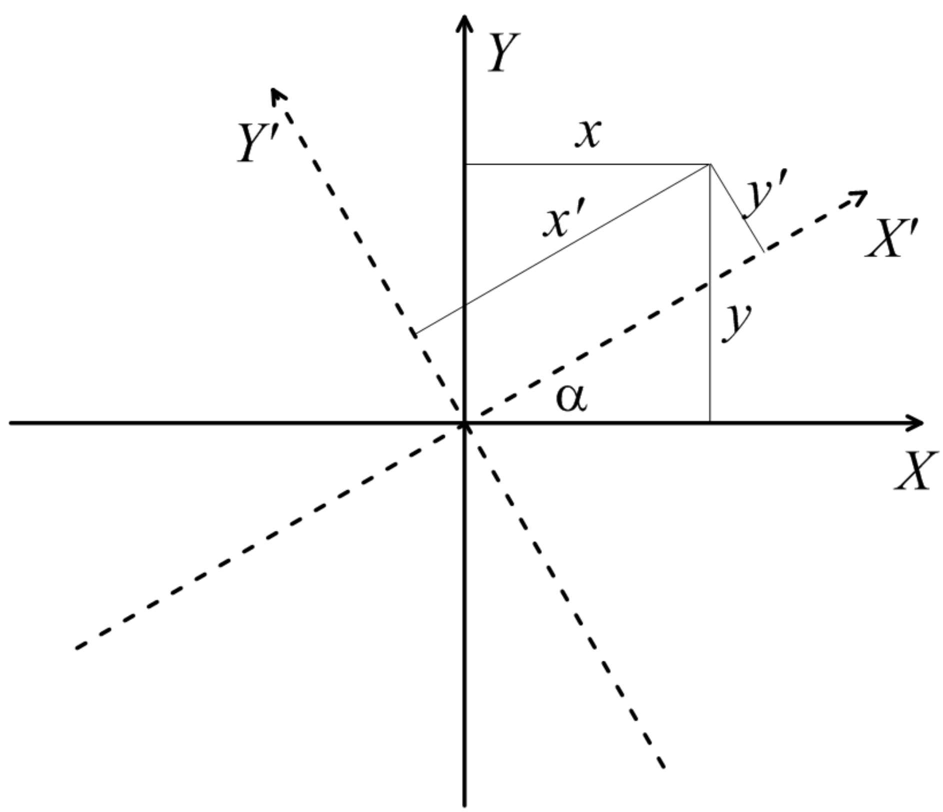
or

$$(x', y') = (x\cos\alpha + y\sin\alpha,\ y\cos\alpha - x\sin\alpha),$$

see Class 4.

Solution

Consider point (x, y) in a Cartesian coordinate system (*X*, *Y*) which has coordinates (x', y') in the Cartesian coordinate system (*X'*, *Y'*) rotated counterclockwise by angle α with respect to (*X*, *Y*), see plot below.



From geometric considerations:

$$x' = \frac{x}{\cos\alpha} + (y - x\tan\alpha)\sin\alpha,$$

or

$$x' = y\sin\alpha + x\left(\frac{1}{\cos\alpha} - \frac{\sin^2\alpha}{\cos\alpha}\right) = y\sin\alpha + x\frac{1-\sin^2\alpha}{\cos\alpha},$$

resulting in

$$x' = x\cos\alpha + y\sin\alpha.$$

Analogously,

$$y' = (y - x\tan\alpha)\cos\alpha,$$

which results in

$$y' = y\cos\alpha - x\sin\alpha.$$

2. Vector in rotated Cartesian plane coordinates

Consider vector $\vec{\mathrm{b}}$ in a Cartesian, (*X*, *Y*), system with orthogonal basis $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$ (unit vectors, see Class 4):

$$\vec{\mathrm{b}} = \mathrm{b}_x\hat{\mathrm{i}} + \mathrm{b}_y\hat{\mathrm{j}}.$$

Now take another 2-D Cartesian system, (*X′*, *Y′*), rotated relative to the (*X*, *Y*) system by angle θ counterclockwise. In terms of unit vectors (orthogonal basis) $\hat{\mathrm{I}}$, $\hat{\mathrm{J}}$ in the rotated system (*X′*, *Y′*), vector $\vec{\mathrm{b}}$ may be expressed as

$$\vec{\mathrm{b}} = \mathrm{b}_{x'}\hat{\mathrm{I}} + \mathrm{b}_{y'}\hat{\mathrm{J}}.$$

Let us express components of vector $\vec{\mathrm{b}}$ in these new coordinates. Using rules of coordinate conversion (see p. **1**), we have

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = y\cos\theta - x\sin\theta.$$

Now replace coordinates by corresponding vector projections on the coordinate axes, i.e., take

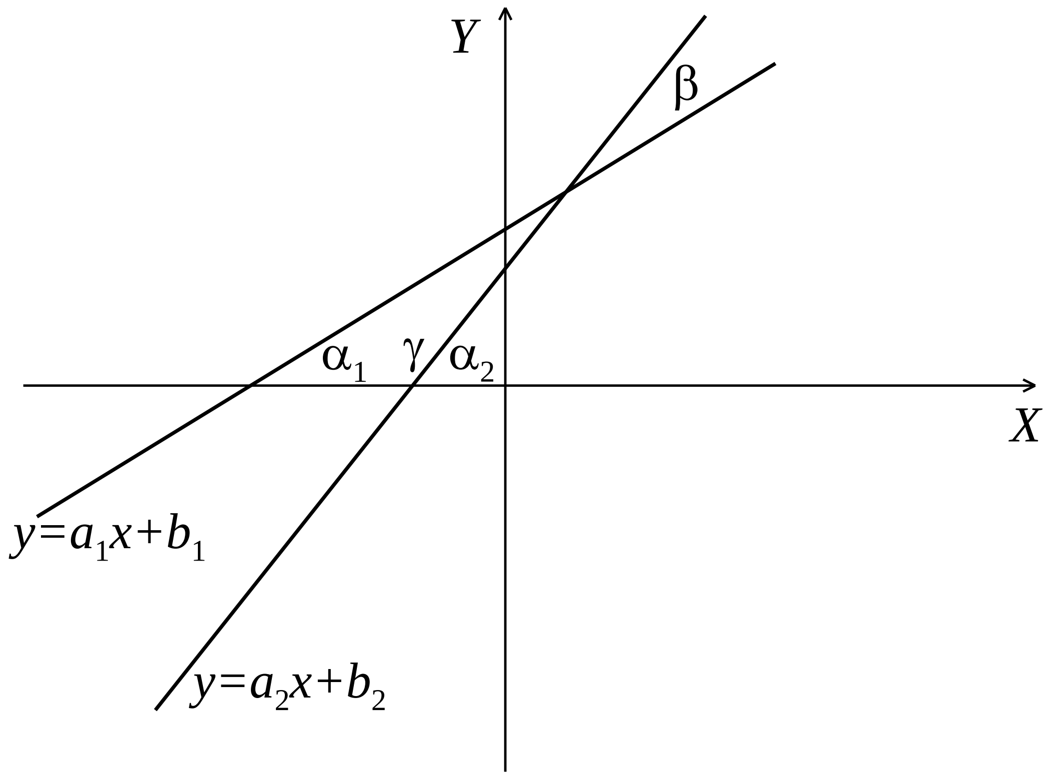
$$\mathrm{b}_{x'} = x',\ \mathrm{b}_{y'} = y',\ \mathrm{b}_x = x,\ \mathrm{b}_y = y.$$

This will provide

$$\mathrm{b}_{x'} = \mathrm{b}_x\cos\theta + \mathrm{b}_y\sin\theta,\ \mathrm{b}_{y'} = -\mathrm{b}_x\sin\theta + \mathrm{b}_y\cos\theta.$$

3. Angle between two straight lines in 2-D Cartesian system and angle between two vectors

See two intersecting straight lines illustrated in the plot below.



Here, a_1 is the tangent of the angle α_1 between the first, $y = a_1x + b_1$, line and the X axis, and a_2 is the tangent of the angle α_2 between the second, $y = a_2x + b_2$, line and the X axis.

The angle β between $y = a_1x + b_1$ and $y = a_2x + b_2$ may be obtained from two equations based on geometrical considerations:

$$\alpha_1 + \gamma + \beta = \pi \text{ and } \alpha_2 + \gamma = \pi,$$

which provides

$$\beta = \alpha_2 - \alpha_1,$$

or

$$\tan\beta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{a_2 - a_1}{1 + a_1a_2}.$$

This means that the lines are parallel when

$$a_2 = a_1 \text{ (in this case, } \beta = \tan\beta = 0\text{)},$$

and the lines are perpendicular ($\tan\beta = \infty$, $\beta = \pi / 2$) when

$$1 + a_1a_2 = 0, \text{ or } a_1a_2 = -1, \text{ or } a_2 = -1/a_1.$$

In terms of the general form of line presentation in Cartesian coordinates ($Ax + By + C = 0$, see Class 4), equation $y = a_1x + b_1$ would correspond to

$$A_1x + B_1y + C_1 = 0 \text{ with } A_1 = a_1,\ B_1 = -1, \text{ and } C_1 = b_1,$$

while equation $y = a_2x + b_2$ would correspond to

$$A_2x + B_2y + C_2 = 0 \text{ with } A_2 = a_2,\ B_2 = -1, \text{ and } C_2 = b_2,$$

so the angle between the lines will be given by

$$\tan\beta = \frac{A_2 - A_1}{1 + A_1A_2}.$$

Condition for the lines to be parallel ($\beta = 0,\ \tan\beta = 0$):

$$A_2 = A_1,$$

Condition for the lines to be perpendicular ($\tan\beta = \infty,\ \beta = \pi/2$)

$$1 + A_1A_2 = 0, \text{ or } A_1A_2 = -1, \text{ or } A_2 = -1/A_1.$$

One may use the above result to derive an expression for angle between two vectors on the plane and establish conditions for two vectors on the plane to be mutually parallel or perpendicular. Consider two 2-D vectors:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} \text{ and } \vec{b} = b_x\hat{i} + b_y\hat{j}.$$

These vectors are aligned, respectively, with straight lines represented by

$$y = \frac{a_y}{a_x}x \text{ and } y = \frac{b_y}{b_x}x.$$

The angle β between these lines (and thus, between the vectors) will be given by

$$\tan\beta = \frac{\frac{b_y}{b_x} - \frac{a_y}{a_x}}{1 + \frac{a_y}{a_x}\frac{b_y}{b_x}} = \frac{a_xb_y - a_yb_x}{a_xb_x + a_yb_y}.$$

Therefore, the vectors will be mutually parallel if

$$a_xb_y - a_yb_x = 0 \text{ or } \frac{a_x}{a_y} = \frac{b_x}{b_y},$$

and mutually perpendicular when

$$a_xb_x + a_yb_y = 0 \text{ or } \frac{a_x}{a_y} = -\frac{b_y}{b_x}.$$

For instance, vectors

$$\vec{a} = 6\hat{i} + 2\hat{j} \text{ and } \vec{b} = 3\hat{i} + 1\hat{j}$$

would be parallel, and vectors

$$\vec{a} = 6\hat{i} + 2\hat{j} \text{ and } \vec{b} = 3\hat{i} - 9\hat{j}$$

would be normal (perpendicular) to each other.

4. Circle in 2-D Cartesian and polar coordinate systems

Show that a circle of radius *c*, which in Cartesian coordinates (*x*, *y*) on a plane appears as $(x-a)^2+(y-b)^2=c^2$, where (*a*, *b*) are the coordinates of the circle center, in 2-D polar coordinates *r* and θ is presented by

$$r^2-2rr_0\cos(\theta-\theta_0)+r_0^{\,2}=c^2 .$$

Solution

Express coordinates *x* and *y* through *r* and θ as

$$x=r\cos\theta \text{ and } y=r\sin\theta .$$

Introduce polar coordinates of the circle center r_0 and θ_0, which would be related to their Cartesian counterparts *a* and *b* as

$$a=r_0\cos\theta_0 \text{ and } b=r_0\sin\theta_0 .$$

Bring these expressions together into

$$(x-a)^2+(y-b)^2=c^2$$

to get

$$(r\cos\theta-r_0\cos\theta_0)^2+(r\sin\theta-r_0\sin\theta_0)^2=r^2-2rr_0(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0)+r_0^{\,2}$$

$$=r^2-2rr_0\cos(\theta-\theta_0)+r_0^{\,2}=c^2 .$$