Lecture 34. November 16, 2016

Topics: Horizontal momentum equations in the isobaric coordinates. Total derivative in the isobaric coordinates. Continuity and thermodynamic energy equations in the isobaric coordinates.

Reading: Holton and Hakim Section 3.1.

1. Horizontal momentum equations

The horizontal momentum equations are approximate equations of horizontal motion that may be obtained by scale analysis from the general-form momentum equation,

$$\frac{d\mathbf{U}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} - 2\mathbf{\Omega}\times\mathbf{U} + \mathbf{F}_r .$$

by retaining only main (that is, representative of the synoptic-scale motions) terms in the individual equations for x and y components of momentum balance (see Class 29). These horizontal momentum equations appear as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv ,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu .$$

where $f = 2\Omega\sin\varphi$ is the Coriolis parameter, and u and v are, respectively, x and y components of the horizontal wind velocity vector $\mathbf{V} = \mathbf{i}u + \mathbf{j}v$.

In vector form, we can write these two equations as a single vector equation,

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} = -\frac{1}{\rho}\nabla p ,$$

where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$ is the horizontal del operator (in this case, we consider changes of the pressure field only in the horizontal *X*-*Y* plane). **Note** that in this case we assume that horizontal derivatives are taken with z held constant.

2. Expression of pressure gradient in isobaric coordinates

Pressure as a vertical coordinate was considered in Class 25, where the following relationships between the components of pressure gradient force in x, y, z and x, y, p coordinate systems were obtained:

$$\left(\partial p/\partial x\right)_z = -\left(\partial p/\partial z\right)_x\left(\partial z/\partial x\right)_p \text{ and } \left(\partial p/\partial y\right)_z = -\left(\partial p/\partial z\right)_y\left(\partial z/\partial y\right)_p .$$

Using the hydrostatic approximation and the approximate form of the geopotential Φ change with height,

$$\partial p/\partial z = -\rho g \text{ and } \partial\Phi/\partial z = g ,$$

we come to

$$(1/\rho)\left(\partial p/\partial x\right)_z = \left(\partial \Phi/\partial x\right)_p \text{ and } (1/\rho)\left(\partial p/\partial y\right)_z = \left(\partial \Phi/\partial y\right)_p ,$$

so we can write down the horizontal momentum equation (p. **1**) in vector form as

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} = -\nabla_p\Phi ,$$

where $\nabla_p = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$ is the horizontal gradient operator applied along isobaric surfaces (that is, with pressure p held constant), and $\mathbf{V} = \mathbf{i}u + \mathbf{j}v$ is horizontal velocity vector in isobaric (pressure) coordinates, where u and v are, respectively, components of velocity aligned with x and y directions in the isobaric coordinates.

Recalling that geostrophic balance in the equations of horizontal motion is introduced as a state of the flow, in which Coriolis and pressure-gradient forces balance each other (Class 29), we obtain the following isobaric-coordinate form of the geostrophic relationship:

$$f\mathbf{k}\times\mathbf{V}_g = -\nabla_p\Phi ,$$

or

$$\mathbf{V}_g = \frac{1}{f}(\mathbf{k}\times\nabla_p\Phi) ,$$

where $\mathbf{V}_g = \mathbf{i}u_g + \mathbf{j}v_g$ is the geostrophic wind in the isobaric coordinates. This relationship means that, as long as Coriolis parameter f may be considered constant, geostrophic wind field on the isobaric surface is locally non-divergent:

$$\nabla_p\cdot\mathbf{V}_g = \frac{1}{f}\nabla_p\cdot(\mathbf{k}\times\nabla_p\Phi) = 0 ,$$

(please be able to demonstrate it yourself).

3. Total (substantial) derivative in isobaric coordinates

In the isobaric coordinates, the *total* (also called *substantial* or *material*) derivative with respect to time, $\frac{d}{dt}$, acquires the form:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p} .$$

The above formula is obtained based on the same considerations as the expression of the total derivative in the Cartesian coordinates:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} .$$

Denoting $\frac{dp}{dt} \equiv \omega$, and using the conventional notation for the horizontal velocity components, $u \equiv \frac{dx}{dt}$ and $v \equiv \frac{dy}{dt}$, we come to

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}.$$

Note that $\omega \equiv \frac{dp}{dt}$ is the isobaric-system counterpart of the Cartesian vertical velocity $w \equiv \frac{dz}{dt}$, so the complete velocity vector in the isobaric coordinates is given by $\mathbf{U} = (\mathbf{V}, \omega)$.

4. Continuity equation in the isobaric coordinates

In the Cartesian coordinates, the continuity equation (that is an expression for the conservation of mass in a moving fluid) is written in the following forms previously considered throughout the course:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = 0, \quad \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{U} = 0.$$

To derive the expression for conservation of mass in the isobaric coordinate system, where the coordinates are *x*, *y*, *p*, and *t*, we recall that *p* may be related to *z* through the hydrostatic balance equation:

$$\partial p / \partial z = -\rho g.$$

We may thus write down the mass conservation principle for a moving fluid parcel with mass

$$\delta M = \rho \delta V = \rho \delta x \delta y \delta z = -\delta x \delta y \frac{\delta p}{g} = -\frac{\delta x \delta y \delta p}{g}$$

as

$$\frac{1}{\delta M}\frac{d}{dt}\delta M = \frac{g}{\delta x \delta y \delta p}\frac{d}{dt}\frac{\delta x \delta y \delta p}{g} = \frac{1}{\delta x \delta y \delta p}\frac{d}{dt}\delta x \delta y \delta p = 0 \text{ (the mass } \delta M \text{ is conserved!).}$$

Performing differentiation in the third term of the above relationship:

$$\frac{1}{\delta x \delta y \delta p}\frac{d}{dt}\delta x \delta y \delta p = \frac{1}{\delta x}\frac{d}{dt}\delta x + \frac{1}{\delta y}\frac{d}{dt}\delta y + \frac{1}{\delta p}\frac{d}{dt}\delta p = \frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0,$$

and recalling that the vertical velocity ω in the isobaric (*x*, *y*, *p*, *t*) coordinate system is defined as $\omega \equiv \frac{dp}{dt}$ (see p. **3**), we come to

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0,$$

which in the limit of $\delta x, \delta y, \delta p \to 0$ provides the continuity equation in the isobaric coordinate system,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0.$$

Note that in the above equation, the partial derivatives in *x* and *y* directions are evaluated at constant pressure.

5. Heat balance (thermodynamic energy) equation in the isobaric coordinates

In the Cartesian coordinates, the heat balance (thermal energy) equation – that is an expression of the first law of thermodynamic in a moving fluid – reads (in one of its previously discussed forms, see Class 33):

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = J \ ,$$

where the total derivative with respect to time is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \ .$$

Substituting into the thermal energy equation the expressions of the total derivatives of temperature and pressure in the isobaric coordinates,

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p} \text{ and } \omega = \frac{dp}{dt},$$

and using the gas law that provides

$$\alpha = \frac{1}{\rho} = \frac{RT}{p},$$

we arrive at the following heat balance equation in the isobaric coordinates:

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \omega\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p} \ .$$

From the definition of potential temperature θ (Class 33),

$$\theta = T(p_0 / p)^{R/c_p},$$

it follows that

$$\frac{\partial \ln\theta}{\partial p} = \frac{\partial \ln T}{\partial p} - \frac{R}{c_p}\frac{\partial \ln p}{\partial p},$$

which may be rewritten as

$$\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{\partial T}{\partial p} - \frac{RT}{c_p p}\frac{\partial p}{\partial p},$$

or

$$\frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial\theta}{\partial p} \equiv S_p,$$

where S_p may be interpreted as the parameter of hydrostatic stability in the isobaric coordinate system. Because

$$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{dT}{dz} + \frac{g}{c_p} = \Gamma_d - \Gamma,$$

see Class 33, we may express stability parameter S_p as

$$S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial z}\frac{\partial z}{\partial p} = \frac{T}{\theta}\frac{\partial\theta}{\partial z}\frac{1}{\rho g} = \frac{\Gamma_d - \Gamma}{\rho g},$$

so S_p is positive when $\Gamma < \Gamma_d$ (subadiabatic regime, $\frac{\partial\theta}{\partial z} > 0$; the atmosphere is stable) and S_p is negative when $\Gamma > \Gamma_d$ (superadiabatic regime, $\frac{\partial\theta}{\partial z} < 0$; the atmosphere is unstable) Under neutral conditions ($\Gamma = \Gamma_d$; $\frac{\partial\theta}{\partial z} = 0$), parameter S_p goes to zero.

The heat balance equation in the isobaric coordinates thus may be written in terms of S_p and ω as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = \frac{J}{c_p},$$

or

$$\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla_p T - S_p\omega = \frac{J}{c_p},$$

where $\nabla_p = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$ is applied along the isobaric surfaces, see p. **2**.