Lecture 29. November 4, 2016

Topics: Energy conservation principle applied to a fluid element. Energy balance equation. Kinetic energy balance. Mechanical energy equation. Thermodynamic energy equation.

Reading: Chapter 2 of Holton and Hakim.

1. Energy conservation principle applied to a fluid element

A control volume δV (see Class 31) containing a specified fluid with density ρ (thus, the mass of the fluid element is given by $\delta m = \rho \delta V$) may also be regarded as a thermodynamic system.

The *total energy E* of this volume is introduced as the sum of the *internal energy* (that is kinetic energy of the individual molecules in the volume) and the *kinetic energy* of the macroscopic motion of the volume:

$$E = \rho\left(e + \frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\delta V ,$$

where e is the *internal energy* per unit mass of fluid in the control volume and $\frac{1}{2}\mathbf{U}\cdot\mathbf{U}$ is the *kinetic energy* per unit mass of the fluid. The internal energy per unit mass of the fluid is given by $e = c_v T$, where c_v is the specific heat at constant volume (for dry atmospheric air: $c_v = 717$ J kg^{-1} K^{-1}), see the Thermodynamics class.

The *energy conservation principle* for the considered fluid element may be formulated in the following way: the rate of change of the total energy of the element is equal to the rate of *diabatic heating* of the fluid in the control volume plus the rate at which work is done on the element by external forces.

Diabatic heating rate J is amount of thermal energy received (positive heating) or given away (negative heating that is cooling) by a unit mass of the fluid element per unit time. Possible heating mechanisms for a moving fluid element in the atmosphere are radiation, conduction, phase transformations. The diabatic heating rate of the fluid element δV is thus given by the product $\rho J \delta V$.

2. Work done by external forces

In the previous sections of the course we have considered the following two types of forces acting on a moving fluid element: surface forces (pressure-gradient and viscous forces) and body forces (gravity and Coriolis forces). We have mentioned also before that the effects of (molecular) viscosity can be neglected in most atmospheric motions.

The resulting rate of work of the pressure force acting along x axis on the volume $\delta V = \delta x \delta y \delta z$ is given by

$$(pu)\big|_x \ \delta y \ \delta z - (pu)\big|_{x+\delta x} \ \delta y \ \delta z .$$

Note that the work is considered positive if the action is directed **into** the volume (see Fig. 2.7 in the textbook).

Expanding product pu (this quantity has the meaning of work per unit time per unit area, or power per unit area that corresponds to the *energy flux*, see considerations regarding the general notion of flux in previous classes) in a Taylor series, we obtain:

$$(pu)\big|_x \ \delta y\ \delta z - (pu)\big|_{x+\delta x}\ \delta y\ \delta z = \left[(pu)\big|_x - (pu)\big|_{x+\delta x}\right]\ \delta y\ \delta z = -\frac{\partial pu}{\partial x}\ \delta V\ .$$

Applying the same procedures to the pressure force work along *y* and *z* directions, we come to

$$(pv)\big|_y \ \delta x\ \delta z - (pv)\big|_{y+\delta y}\ \delta x\ \delta z = \left[(pv)\big|_y - (pv)\big|_{y+\delta y}\right]\ \delta x\ \delta z = -\frac{\partial pv}{\partial y}\ \delta V\ ,$$

$$(pw)\big|_z \ \delta x\ \delta y - (pw)\big|_{z+\delta z}\ \delta x\ \delta y = \left[(pw)\big|_z - (pw)\big|_{z+\delta z}\right]\ \delta x\ \delta y = -\frac{\partial pw}{\partial z}\ \delta V\ .$$

By recalling rules of vector differentiation, the total rate of work by pressure force can be expressed as

$$-\frac{\partial pu}{\partial x}\delta V - \frac{\partial pv}{\partial y}\ \delta V - \frac{\partial pw}{\partial z}\ \delta V = -\nabla\cdot(p\mathbf{U})\delta V\ .$$

There are contributions by two other forces we need to consider: the work by the Coriolis force and the work by the gravity force.

The *work by the Coriolis force* on the moving fluid element in the Earth's atmosphere is *zero* because this force is directed perpendicular to the velocity vector:

$$\rho\delta V(-2\mathbf{\Omega}\times\mathbf{U})\cdot\mathbf{U} = 0\,.$$

The work rate of the *gravity force* is defined as product of mass, acceleration of gravity (this gives gravity force) and distance per unit time (distance per unit time is velocity):

$$\rho\ (\mathbf{g}\cdot\mathbf{U})\ \delta V\ ,$$

which in the component form corresponds to

$$-\rho g w \delta V\ .$$

3. Energy balance of unit volume of the fluid

Putting together all work and heating (thermal energy) rate terms considered above, we come to the following equation that presents the energy balance of the fluid element δV :

$$\frac{dE}{dt} = \frac{d}{dt}\left[\rho\left(e+\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\delta V\right] = -\nabla\cdot(p\mathbf{U})\delta V + \rho\ (\mathbf{g}\cdot\mathbf{U})\delta V + \rho J\,\delta V\ .$$

However, the mass of the moving fluid element is conserved, so $\frac{d}{dt}\rho\delta V = 0$, see Class 31. Thus one can write down the above equation in the following form representing the balance of energy per unit volume:

$$\rho\frac{d}{dt}\left(e+\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = \rho\frac{de}{dt} + \rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - p\nabla\cdot\mathbf{U} + \rho\mathbf{g}\cdot\mathbf{U} + \rho J\ .$$

4. Kinetic energy balance

Equation presenting the balance of kinetic energy per unit volume can be derived from the general momentum balance equation (see Class 27) with neglected viscous friction force:

$$\frac{d\mathbf{U}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} - 2\mathbf{\Omega}\times\mathbf{U},$$

or

$$\rho\frac{d\mathbf{U}}{dt} = -\nabla p + \rho\mathbf{g} - 2\rho\mathbf{\Omega}\times\mathbf{U}.$$

We take dot products of the both sides of the above equation with velocity vector $\mathbf{U}$. Taking into consideration that $[\mathbf{\Omega}\times\mathbf{U}]\cdot\mathbf{U}$ =0, this results in the following equation:

$$\rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p + \rho\mathbf{g}\cdot\mathbf{U},$$

which expresses the balance of kinetic energy per unit volume of the fluid. **Note** that there is no contribution from the Coriolis force there.

Consequently, the *kinetic energy balance* per unit mass of the fluid is given by

$$\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\frac{1}{\rho}\mathbf{U}\cdot\nabla p + \mathbf{g}\cdot\mathbf{U}.$$

5. Mechanical energy balance

Mechanical energy is defined as the sum of kinetic and potential energies of the fluid. Using this definition, and taking into account that potential energy is represented in our case by only the energy associated with gravity and that

$$\mathbf{g}\cdot\mathbf{U} = \mathbf{g}\cdot\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{g}\cdot\mathbf{r}}{dt},$$

we can reformulate the kinetic energy balance equation (p. **4**) as

$$\rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U} - \mathbf{g}\cdot\mathbf{r}\right) = -\mathbf{U}\cdot\nabla p,$$

and treat it as a *mechanical energy balance* equation. In terms of geopotential Φ, using relationships $\mathbf{g} = -g\mathbf{k}$, $g = |\mathbf{g}|$, and $d\Phi = gdz$, the above equation may be written as

$$\rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U} + \Phi\right) = -\mathbf{U}\cdot\nabla p,$$

providing another form of the mechanical energy balance equation.

6. Thermodynamic energy equation

Subtracting the kinetic energy balance equation in the form:

$$\rho \frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p + \rho\mathbf{g}\cdot\mathbf{U}\,,$$

(see p. **4**), from the total energy balance equation,

$$\rho\frac{de}{dt} + \rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - p\nabla\cdot\mathbf{U} + \rho\mathbf{g}\cdot\mathbf{U} + \rho J\,,$$

(see p. **3**), we come to the balance equation for the internal (thermal) energy per unit volume

$$\rho\frac{de}{dt} = -p\nabla\cdot\mathbf{U} + \rho J\,.$$

Taking into account that $e = c_v T$ (see p. **1**) and using the continuity equation in the form: $\nabla\cdot\mathbf{U} = -\frac{1}{\rho}\frac{d\rho}{dt}$ (see Class 31), we first obtain

$$c_v\frac{dT}{dt} = -\frac{p}{\rho}\nabla\cdot\mathbf{U} + J = \frac{p}{\rho^2}\frac{d\rho}{dt} + J = -p\frac{d(1/\rho)}{dt} + J\,,$$

which leads to the traditional form of the *thermodynamic energy (balance) equation*:

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J\,,$$

where $\alpha = 1/\rho$ is the specific volume of the fluid (air).

Note that the derived expression is another form of the *First Law of Thermodynamics* (FLT, see your Thermodynamics class notes).

Indeed, applying $dq = c_v dT + p d\alpha$ (which is one of the FLT formulations commonly used in thermodynamics), where dq is the differential increment of thermal energy added to the unit mass of substance (which is *moving* air in our case), and considering all changes as happening in time (per unit time), one arrives at the FLT for the unit mass of moving fluid in the form:

$$J = \frac{dq}{dt} = c_v\frac{dT}{dt} + p\frac{d\alpha}{dt}\,.$$