**Lecture 28.** November 2, 2016

**Topics:** Conservation of mass in a moving fluid: Eulerian derivation. Continuity equation and divergence theorem. Lagrangian derivation of the continuity equation. Scale analysis of the continuity equation; anelastic and incompressibility approximations. Boussinesq approximation.

**Reading:** Chapter 2 of Holton and Hakim.

## 1. Conservation of mass in a moving fluid: Eulerian derivation

Mass conservation law for a fluid element (control volume) in a moving fluid may be considered based either in Eulerian (the fluid element is fixed in a coordinate frame) or Lagrangian (the fluid element follows the fluid motion) frames, see Class 26.

We consider a fluid volume element $\delta x \delta y \delta z$ fixed in a right-hand Cartesian coordinate frame (see Fig. 2.5 in the textbook). The volume element is centered against the location $x, y, z$. Let us look at the balance of mass along the *x* axis in this volume element.

The quantity $\rho(x, y, z)\, u(x, y, z)$, where $\rho$ is the density of the fluid and $u$ (as usually denoted) is the *x* component of the flow velocity vector (fluid motion), represents the mass transported per unit area normal to the *x* axis per unit time at the location $x, y, z$. This quantity is the *x* component of the so-called *mass flux*.

We may also consider the mass flux at $x - \frac{\delta x}{2}, y, z$ (that is the left facet of the volume shown in Fig. 2.5): $\rho(x - \delta x, y, z)\, u(x - \delta x, y, z)$ that represents the mass *inflow rate* per unit area to the control volume through the left facet. In the linear approximation:

$$\rho(x - \delta x/2, y, z)\, u(x - \delta x/2, y, z) = \rho(x, y, z)\, u(x, y, z) - \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}.$$

The mass flux through the right facet (that is the mass *outflow rate* per unit area from the control volume) at $x + \frac{\delta x}{2}, y, z$ is given in the linear approximation by

$$\rho(x + \delta x/2, y, z)\, u(x + \delta x/2, y, z) = \rho(x, y, z)\, u(x, y, z) + \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}.$$

Thus, the *net mass flow rate* (that is mass per unit time) along *x* axis into the control volume is

$$\rho(x - \delta x/2, y, z)\, u(x - \delta x/2, y, z)\ \delta y\ \delta z - \rho(x + \delta x/2, y, z)\, u(x + \delta x/2, y, z)\ \delta y\ \delta z =$$

$$-\frac{\partial \rho u}{\partial x}\delta x \delta y \delta z\ .$$

Considering mass inflow and outflow rates along the other two directions, *y* and *z*, one may obtain similar expressions for the corresponding net mass flow rates:

$$\rho(x, y-\delta y/2, z)\ u(x, y-\delta y/2, z)\ \delta x\ \delta z\ -\rho(x, y+\delta y/2, z)\ u(x, y+\delta y/2, z)\ \delta x\ \delta z =$$

$$-\frac{\partial \rho v}{\partial y}\delta x \delta y \delta z\ ,$$

where $v$ is the $y$ component of the velocity, and

$$\rho(x, y, z-\delta z/2)\ u(x, y, z-\delta z/2)\ \delta x\ \delta y\ -\rho(x, y, z+\delta z/2)\ u(x, y, z+\delta z/2)\ \delta x\ \delta y =$$

$$-\frac{\partial \rho w}{\partial z}\delta x \delta y \delta z\ ,$$

where $w$ is the $z$ component of the velocity.

The *total* net mass flow rate into the control volume is therefore:

$$-\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}\right)\delta x \delta y \delta z\ ,$$

which corresponds to the following mass per unit volume (that is, density) rate:

$$-\frac{\partial \rho u}{\partial x}-\frac{\partial \rho v}{\partial y}-\frac{\partial \rho w}{\partial z}\ .$$

## 2. Continuity equation and divergence theorem

The above expression represents the change of mass of fluid per unit volume per unit time at $x$, $y$, $z$ due to the mass transport to this location (mass flux) from all three coordinate directions. We see that the considered mass per unit volume (density) change associated with the mass flux must be (the fluid does not disappear in the control volume!) equal to the local increase of density per unit time, $\frac{\partial \rho}{\partial t}$. Accordingly, the conservation of mass locally in the fluid is presented by

$$\frac{\partial \rho}{\partial t}=-\frac{\partial \rho u}{\partial x}-\frac{\partial \rho v}{\partial y}-\frac{\partial \rho w}{\partial z}\ ,$$

which may be rewritten as

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0\ ,$$

and eventually, in vector form, as

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathbf{U}=0\ ,$$

where velocity vector $\mathbf{U}$ has components $u, v,$ and $w$ in $x$, $y$, and $z$ directions, respectively. The obtained relationship is called the *continuity equation*.

The continuity equation presented above has been previously derived in Class 15 by applying the divergence theorem of vector calculus. This theorem states that for a continuously differentiable vector field **B** defined within volume $V$, which has a (piecewise smooth) boundary $A$, the following equality holds:

$$\int_V \nabla\cdot\mathbf{B}\, dV = \int_A \mathbf{B}\cdot\mathbf{n}\, dA \equiv \int_A \mathbf{B}\cdot d\mathbf{A}\,,$$

where **n** is the outward pointing local unit vector normal to the boundary. If we take *V* as a fixed volume in the moving fluid and **B** as a vector of mass flux (this would be also a vector of momentum per unit volume), i.e. substitute $\mathbf{B} = \rho\mathbf{U}$, we will have according to the divergence theorem (see Class 15):

$$\int_A \rho\mathbf{U}\cdot d\mathbf{A} = \int_V \nabla\cdot\rho\mathbf{U}\, dV\,,$$

which means that integral of the flux divergence over volume *V* is equal to the integral of the flux over a surface *A* that surrounds the volume *V*. The integral mass flux, $\int_A \rho\mathbf{U}\cdot d\mathbf{A}$, represents the total change of fluid mass in the volume *V* per unit time, and thus

$$-\frac{d}{dt}\int_V \rho\, dV = -\int_V \frac{\partial\rho}{\partial t}\, dV = \int_A \rho\mathbf{U}\cdot d\mathbf{A} = \int_V \nabla\cdot\rho\mathbf{U}\, dV\,,$$

or

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{U}\right) dV = 0\,,$$

which provides the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{U} = 0\,.$$

Using identities $\nabla\cdot\rho\mathbf{U} = \rho\nabla\cdot\mathbf{U} + \mathbf{U}\cdot\nabla\rho$ and $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla$, we come to another form of the continuity equation:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{U} = 0\,.$$

For the incompressible fluid ( $\rho$ =const, so $\frac{d\rho}{dt} = 0$ ), the above equation has the form

$$\nabla\cdot\mathbf{U} = 0\,.$$

In atmospheric dynamics, various continuity equation forms are used depending on scales (mostly vertical) of the considered atmospheric motions.

### 3. Lagrangian derivation of the continuity equation

In 3-D Cartesian coordinates we consider a control volume $\delta V$ of a fixed mass

$$\delta M = \rho\delta V = \rho\delta x\delta y\delta z$$

moving with the fluid. The total change of mass:

$$\frac{1}{\delta M}\frac{d}{dt}\delta M = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d}{dt}\delta V$$

is equal zero due to the fact that the mass of the control volume is conserved following the motion. Because

$$\frac{1}{\delta x}\frac{d}{dt}\delta x = \frac{\delta u}{\delta x},\ \frac{1}{\delta y}\frac{d}{dt}\delta y = \frac{\delta v}{\delta y},\ \text{and}\ \frac{1}{\delta z}\frac{d}{dt}\delta z = \frac{\delta w}{\delta z},$$

we obtain in the limit of $\delta x \to 0, \delta y \to 0, \delta z \to 0$ the following expression:

$$\frac{1}{\delta V}\frac{d}{dt}\delta V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{U}.$$

where $\mathbf{U} = (u, v, w)$ is the velocity vector. Therefore,

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{U} = 0,$$

that looks exactly like one of vector forms of the continuity equation considered in p. **2**.

**4. Scale analysis of the continuity equation; anelastic and incompressibility approximations**

In p. **2**, we considered the following forms of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = 0,$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho = 0,$$

and

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{U} = 0.$$

In order to evaluate the relative significance of different terms of this equation for synoptic-scale atmospheric motions, we will decompose the density field into the standard (hydrostatic) value that depends only on *z* and the deviation from this standard (reference) value:

$$\rho(x, y, z, t) = \rho_r(z) + \rho'(x, y, z, t).$$

The same approach was used in Class 29 to evaluate deviations from the hydrostatic balance in the synoptic-scale equations of motion.

Substituting the above expression in the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U}$ and taking into account that $|\rho'/\rho_r| << 1$ (for synoptic-scale motions: $|\rho'/\rho_r| \sim 10^{-2}$), we will come to the following approximate equality:

$$\frac{1}{\rho_r}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \cdot \nabla \rho'\right) + \frac{w}{\rho_r}\frac{d\rho_r}{dz} + \nabla \cdot \mathbf{U} \approx 0.$$

Now we evaluate individual terms of the approximate form of the continuity equation using the following synoptic-motion scales considered in Class 29:

$L$~$10^3$ km=$10^6$ m as the length scale;

$H$~10 km = $10^4$ m as the depth scale;

$U$~10 m s$^{-1}$ as the horizontal velocity scale;

$W$~1 cm s$^{-1}$ =$10^{-2}$ m s$^{-1}$ as the vertical velocity scale;

$L/U$~$10^5$ s as the time scale.

The evaluation provides

$$\frac{1}{\rho_r}\left(\frac{\partial\rho'}{\partial t}+\mathbf{U}\cdot\nabla\rho'\right)\sim\left|\frac{\rho'}{\rho_r}\right|\frac{U}{L}\sim 10^{-7}\mathrm{s}^{-1},\ \frac{w}{\rho_r}\frac{d\rho_r}{dz}\sim\frac{W}{H}\sim 10^{-6}\mathrm{s}^{-1},\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\sim 0.1\frac{U}{L}\sim 10^{-6}\mathrm{s}^{-1},\ \frac{\partial w}{\partial z}\sim\frac{W}{H}\sim 10^{-6}\mathrm{s}^{-1}.$$

Therefore, in the synoptic-scale motions, the mass conservation (represented by the continuity equation)

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}=0,$$

can be approximately expressed as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{w}{\rho_r}\frac{d\rho_r}{dz}=0,$$

or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+w\frac{d\ln\rho_r}{dz}=0,$$

which in vector form may be written as

$$\nabla\cdot\rho_r\mathbf{U}=0.$$

The resulting approximation of the mass conservation is called the *anelastic* approximation.

This approximation may be compared with the so-called *incompressibility* approximation (see p. **2**),

$$\nabla\cdot\mathbf{U}=0,$$

widely used in the environmental fluid dynamics. It results from applying assumption/condition $\frac{d\rho}{dt}=0$ in $\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{U}=0$, which implies that density of the parcel following the motion does not change (remains constant). The anelastic approximation, on the other hand, takes into account compressibility (in terms of the vertical change of density) associated with the vertical motion.

**5. Boussinesq approximation**

Under conditions when horizontal scales of motion are about or smaller than synoptic and the vertical scale of motion is relatively small (understood as being significantly smaller than the depth of the troposphere), the density changes following the motion may be also assumed small (compared to overall density change throughout the troposphere), so the equations of motion plus the continuity equation (equations of atmospheric dynamics) may be reduced to a simplified form that is called the Boussinesq approximate form (or just the *Boussinesq approximation*) of these equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p}{\partial x}+fv-2\Omega w\cos\varphi+F_{rx},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p}{\partial y}-fu+F_{ry},$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_c}\frac{\partial p}{\partial z}-g\frac{\rho}{\rho_c}+2\Omega u\cos\varphi+F_{rz},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

where density is replaced by a constant reference value $\rho_c$ everywhere except for the gravity (buoyancy) term in the vertical momentum equation. **Note** that $\rho_c \neq \rho_r(z)$, which is the reference (standard) density considered in the scale analysis of synoptic-scale vertical momentum equation presented in Class 29. However, the reference-state density in the atmosphere under some conditions can be assumed constant.