Lecture 22. October 17, 2016

Topics: Examples of motion involving Coriolis force. Missile trajectory. Constant angular momentum (inertial) oscillation. Radius and period of constant angular momentum (inertial) oscillation.

Reading: Section 1.3 of Holton and Hakim.

1. Missile trajectory

In Class 21 we considered components of the Coriolis force (acceleration) that arises in the geocentric frame as reaction to the accelerated motion of the frame attached to the rotating Earth. Let us look at the following example of motion affected by Coriolis force.

Suppose that a ballistic missile is fired eastward at 43ºN latitude (which corresponds to $f \equiv 2\Omega \sin\varphi \simeq 10^{-4}\,\text{rad s}^{-1}$). If the missile travels 1000 km at a horizontal (zonal) speed $u_0 = 1000\ \text{m s}^{-1}$, by how much would the missile get deflected from its eastward path by the Coriolis force?

Solution

From $\dfrac{dv}{dt} = -2\Omega u \sin\varphi$ (see Class 21), we obtain, by integration in time (keeping in mind that v_0, the initial missile's velocity in *y* direction, is zero, and the deflection is assumed to be small, so $\sin\varphi$ may be taken approximately constant while missile travels):

$$v = -2\Omega \sin\varphi u_0 t = -fu_0 t\,.$$

We now integrate v over time to get

$$y - y_0 = \int_0^t \frac{dy}{dt}dt = \int_0^t vdt = -\int_0^t fu_0 tdt = -fu_0 \int_0^t tdt = -fu_0 \frac{t^2}{2}\,,$$

where the time of the flight, t, is equal to the traveled zonal distance $L = 1000\ \text{km} = 10^6\ \text{m}$ divided by the zonal speed $u_0 = 1000\ \text{m s}^{-1}$.

In terms of deviation in the *y* direction, $\delta y \equiv y - y_0$:

$$\delta y = -fu_0 \frac{(L/u_0)^2}{2} = -\frac{fL^2}{2u_0}\,,$$

that is

$$\delta y = -\frac{10^{-4}10^{12}}{2 \cdot 10^3} = -0.5 \cdot 10^5\text{m} = -50\ \text{km}\,.$$

Therefore, the missile would be deflected by the Coriolis force $50\ \text{km}$ to the right of its eastward trajectory, i.e. to the south.

2. Constant angular momentum (inertial) oscillation

Let us now look at the problem of horizontal motion in geocentric frame in a more general way. Suppose an object (particle) initially at rest on the Earth on a horizontal frictionless plate, located at the point (x_0, y_0) with latitude φ, is impulsively propelled eastward along the X axis with a speed V at time t=0. Assume that the latitudinal excursion of the particle is sufficiently small, so φ may be taken constant. How to find position (coordinates) of the particle as function of time?

For particle acceleration components along X and Y axes, respectively, we have in this case (see Class 21):

$$\frac{du}{dt} = 2\Omega v \sin\varphi = fv ,$$

$$\frac{dv}{dt} = -2\Omega u \sin\varphi = -fu .$$

Differentiating the first equation with respect to t,

$$\frac{d^2u}{dt^2} = f\frac{dv}{dt} = -f^2 u ,$$

we come to

$$\frac{d^2u}{dt^2} + f^2 u = 0 ,$$

which is a linear ordinary differential equation having the following general solution:

$$u = A_u \sin ft + B_u \cos ft .$$

From the problem statement: $u = V$ at t=0. Thus, $B_u = V$. On the other hand,

$$\frac{du}{dt} = fA_u \cos ft - fB_u \sin ft ,$$

so

$$v = \frac{1}{f}\frac{du}{dt} = A_u \cos ft - B_u \sin ft .$$

Condition $v = 0$ at t=0 then provides $A_u = 0$. Therefore, the velocity components are given by

$$u = V \cos ft ,$$

$$v = \frac{1}{f}\frac{du}{dt} = -V \sin ft .$$

We can integrate these expressions over time from t=0 to t to obtain the coordinates of the particle position as functions of time:

$$x - x_0 = \int_0^t \frac{dx}{dt}dt = \int_0^t udt = \int_0^t V \cos ftdt = \frac{V}{f}\sin ft\Big|_0^t = \frac{V}{f}\sin ft ,$$

$$y - y_0 = \int_0^t \frac{dy}{dt} dt = \int_0^t v dt = -\int_0^t V \sin ft dt = \frac{V}{f} \cos ft \Big|_0^t = \frac{V}{f} \cos ft - \frac{V}{f} = \frac{V}{f} (\cos ft - 1) .$$

One may rewrite the obtained coordinate expressions as

$$(x - x_0)^2 = \left(\frac{V}{f} \right)^2 \sin^2 ft ,$$

$$\left[y - \left(y_0 - \frac{V}{f} \right) \right]^2 = \left(\frac{V}{f} \right)^2 \cos^2 ft ,$$

and combine them into a single equation:

$$(x - x_0)^2 + \left[y - \left(y_0 - \frac{V}{f} \right) \right]^2 = \left(\frac{V}{f} \right)^2 .$$

3. Radius and period of constant angular momentum (inertial) oscillation

In the Northern Hemisphere, where $f > 0$, the obtained equation represents a circle with radius $R = \frac{V}{f}$ centered at $\left(x_0, y_0 - \frac{V}{f} \right)$, see Class 4. In the orbit, the particle moves clockwise (that is *anticyclonically* relative to the direction of the Earth's rotation).

The period τ of the particle motion along such a circular orbit (that is the time needed for a particle to complete one revolution) is given by

$$\tau = \frac{2\pi R}{V} = \frac{2\pi}{f} = \frac{\pi}{\Omega \sin \varphi} .$$

Thus, an object displaced horizontally from its equilibrium position at (in a sense, very close to) the surface of the Earth will oscillate about this equilibrium position with a period that depends on the latitude. At $\varphi = 30$ºN, for instance, this period will be equal to the duration of the day:

$$\tau = \frac{\pi}{\frac{1}{2}\Omega} = \frac{2\pi}{\Omega} = 86400 \text{ s} .$$

The considered oscillations (periodic orbital motions) associated with the conservation of the angular momentum in the geocentric frame (see Class 21) are often (albeit loosely) called the *inertial oscillations* (based on the fact that Coriolis force arises in conjunction with the inertial relative motion of the particle).