Lecture 20. October 12, 2016

Topics: Zonal and radial deflecting forces in geocentric frame. Coriolis force. Coriolis parameter. Vector representation of the Coriolis force.

Reading: Section 1.3 of Holton and Hakim.

1. Angular momentum; zonal deflecting force

Suppose that an object of unit mass (which could be an air parcel of unit mass) moving with zonal velocity u in the atmosphere near the Earth's surface at latitude φ. We introduce a right-hand Cartesian coordinate system and direct X to the east, Y to the north, and Z outward to the horizontal plane tangential to the Earth's surface at the location of the parcel.

Suppose that this parcel is displaced by $\delta\varphi$ in the direction of equator by an impulsive force over the time increment δt (see Fig. 1.7 in Holton and Hakim). The original absolute angular momentum per unit mass (in general terms introduced as $\mathbf{m} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is position vector and $\mathbf{p}$ is momentum per unit mass) of an air parcel having absolute tangential velocity V (due to its rotation with the Earth plus its motion relative to the Earth) with respect to the axis of the Earth's rotation is

$$VR = (\Omega R + u)R = \left(\Omega + \frac{u}{R}\right)R^2 .$$

In the absence of zonal torque, the angular momentum of zonal motion should be conserved after the parcel has been displaced. Therefore, the following equality should hold:

$$\left(\Omega + \frac{u}{R}\right)R^2 = \left(\Omega + \frac{u+\delta u}{R+\delta R}\right)(R+\delta R)^2 ,$$

where δR and δu are, respectively, changes in R and u that compensate for the displacement of the parcel in order to conserve the angular momentum.

Neglecting the second-order terms that contain products of δR and δu, we come to the following expression for the zonal velocity change:

$$\delta u = -2\Omega\delta R - \frac{u}{R}\delta R = -\left(2\Omega + \frac{u}{R}\right)\delta R .$$

Using $R = a\cos\varphi$ (where a is the Earth's radius) and $\delta R = -\delta y \sin\varphi$, we come to

$$\frac{\delta u}{\delta t} = -2\Omega\frac{\delta R}{\delta t} - \frac{u}{R}\frac{\delta R}{\delta t} = 2\Omega\sin\varphi\frac{\delta y}{\delta t} + \frac{u\sin\varphi}{a\cos\varphi}\frac{\delta y}{\delta t} = 2\Omega\sin\varphi\frac{\delta y}{\delta t} + \frac{u\tan\varphi}{a}\frac{\delta y}{\delta t} .$$

Now, take the limit as $\delta t \to 0$ (so $\frac{\delta u}{\delta t}$ will become $\frac{du}{dt}$) and note that $\frac{dy}{dt} = v$, the y (northward) component of velocity. This yields for the zonal deflecting acceleration induced by the latitudinal displacement:

$$\frac{du}{dt} = 2\Omega \sin\varphi \frac{dy}{dt} + \frac{u\tan\varphi}{a}\frac{dy}{dt} = 2\Omega v \sin\varphi + \frac{uv}{a}\tan\varphi\,,$$

where the term $\frac{uv}{a}\tan\varphi$ represents the so-called curvature effect.

Suppose now that the parcel was displaced vertically by $\delta z > 0$ instead of being displaced with respect to the latitude. In this case, we still have

$$\delta u = -2\Omega\delta R - \frac{u}{R}\delta R\,,$$

but in this case $\delta R = \delta z \cos\varphi$, which will eventually result in the following formula for the zonal deflecting acceleration cause by the vertical displacement of the parcel:

$$\frac{du}{dt} = -2\Omega\cos\varphi\frac{dz}{dt} - \frac{u}{a}\frac{dz}{dt} = -2\Omega w \cos\varphi - \frac{uw}{a}\,,$$

where $\frac{dz}{dt} = w$ is the *z* (vertical) velocity component, and $-\frac{uw}{a}$ is another curvature-effect term.

If both displacements (latitudinal and vertical ones) are superimposed, the resulting zonal (in the direction of *u*) acceleration is given by the combination of the two corresponding expressions:

$$\frac{du}{dt} = 2\Omega v \sin\varphi - 2\Omega w \cos\varphi - \frac{uw}{a} + \frac{uv}{a}\tan\varphi\,,$$

where the first two terms on the right-hand side represent the zonal component of the Coriolis force (see p. **3**) and the last two terms are referred to as the curvature(-effect) terms.

The above equation may also be obtained, as described in the textbook (Sect. 1.3.3), by direct differentiation in time of the expression for the zonal angular momentum, $VR = (\Omega R + u)R = \left(\Omega + \frac{u}{R}\right)R^2$, and setting the derivative to zero (because the angular momentum is conserved).

2. Radial deflecting force

Imagine now that an air parcel originally at rest in the atmosphere near the Earth's surface is set in motion in the eastward direction with velocity *u*. Because the parcel is now rotating faster that the Earth (the angular velocity of the parcel will be $\Omega + \frac{u}{R}$ as compared to Ω for a parcel at rest), the centrifugal force acting on the parcel will be increased.

The excess of the centrifugal force over that for a parcel at rest is

$$\left(\Omega + \frac{u}{R}\right)^2 \mathbf{R} - \Omega^2\mathbf{R} = \frac{2\Omega u\mathbf{R}}{R} + \frac{u^2\mathbf{R}}{R^2} = \left(2\Omega u + \frac{u^2}{R}\right)\hat{\mathbf{R}}\,.$$

Two terms in the parentheses on the right-hand side represent *deflecting forces* acting outward along the vector $\mathbf{R}$.

The second of the two terms (it is another curvature-effect term) is typically very small in the atmosphere compared to the first term because $|u| \ll \Omega R$. Due to the same reason, other considered curvature-effect terms (also the ones in p. **1**) are relatively small and can be neglected, in a first approximation, compared to the terms containing the angular velocity of the Earth's rotation.

The remaining deflecting force (acceleration) due to the Earth's rotation,

$$\frac{2\Omega u\mathbf{R}}{R} = 2\Omega u\hat{\mathbf{R}},$$

has two components along y and z directions (note that $R_y = -R\sin\varphi$ and $R_z = R\cos\varphi$, see Fig. 1.8 in Holton and Hakim):

$$\frac{dv}{dt} = \frac{2\Omega u}{R} R_y = -2\Omega u \sin\varphi,$$

$$\frac{dw}{dt} = \frac{2\Omega u}{R} R_z = 2\Omega u \cos\varphi.$$

3. Coriolis force

Considered in p. **1** and p. **2** deflection forces associated with the Earth's rotation are components of the so-called Coriolis force (acceleration). For instance, the radial deflection force in p. **2**,

$$2\Omega u\hat{\mathbf{R}} = (0,\ -2\Omega u\sin\varphi,\ 2\Omega u\cos\varphi),$$

represents a component of the so-called *Coriolis force* per unit mass (or the *Coriolis acceleration*) in the direction of vector $\mathbf{R}$. To summarize:

x component of the Coriolis acceleration of the parcel is (see p. **1**):

$$a_{Cx} = \left(\frac{du}{dt}\right)_{Co} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi,$$

y component of the Coriolis acceleration of the parcel is (see p. **2**):

$$a_{Cy} = \left(\frac{dv}{dt}\right)_{Co} = -2\Omega u\sin\varphi.$$

z component of the Coriolis acceleration of the parcel is (see p. **2**):

$$a_{Cz} = \left(\frac{dw}{dt}\right)_{Co} = 2\Omega u\cos\varphi,$$

where φ is the latitude of the location of the parcel.

The above expressions mean, in particular, that in the Northern Hemisphere, an air particle (parcel) moving eastward is deflected southward and upward by the Coriolis force. Generally, in the Northern Hemisphere, air parcels moving along the Earth's surface are deflected to the right of the direction of motion. **Note** that there is no projection of the Earth's angular velocity vector on the X axis in the considered coordinate system.

4. Coriolis parameter

The quantity $f \equiv 2\Omega \sin\varphi$, which is broadly encountered in meteorological applications, is called the *Coriolis parameter*. So, one can write the *y* component of the Coriolis force (acceleration) caused by the eastward motion of the parcel with velocity *u* as

$$a_{Cy} = \left(\frac{dv}{dt}\right)_{Co} = -fu\,.$$

Analogously, the resulting Coriolis acceleration in the *x* direction may be written as

$$a_{Cx} = \left(\frac{du}{dt}\right)_{Co} = fv - 2\Omega w \cos\varphi\,.$$

5. Expression of Coriolis force in vector form

In the general case, when the parcel has all three components of motion (*x*, *y*, and *z*), the Coriolis force is expressed as twice the vector product of the Earth's angular velocity vector and the parcel velocity vector:

$$\left(\frac{d\mathbf{U}}{dt}\right)_{Co} = -2\mathbf{\Omega} \times \mathbf{U}\,.$$

This expression will be extensively discussed and used in the subsequent sections of the course.

If only horizontal components of wind velocity are considered (with neglecting the vertical component of wind as it often happens in atmospheric dynamics), $\mathbf{V} \equiv (u, v)$ and *w*=0, then

$$a_{Cx} = \left(\frac{du}{dt}\right)_{Co} = fv \text{ and } a_{Cy} = \left(\frac{dv}{dt}\right)_{Co} = -fu\,,$$

which may be combined in vector form as

$$\left(\frac{d\mathbf{V}}{dt}\right)_{Co} = -f\mathbf{k} \times \mathbf{V}\,.$$

Note:

(*i*) effects of Coriolis force are negligible for motions with time scales much shorter than the period of the Earth's rotation, and

(*ii*) this force affects only direction, not the speed of motion (velocity magnitude).