Lecture 19. October 10, 2016

Topics: Apparent forces in the geocentric frame. Centripetal acceleration. Centrifugal force. Gravity force. Geopotential.

Reading: Section 1.3 of Holton and Hakim.

1. Apparent forces in the geocentric frame

In physical mechanics, as was noted in Class 19, one distinguishes between the *inertial* (fixed in space or uniformly moving with respect to a frame fixed in space) and *noninertial* (moving with acceleration) reference frames. In the atmospheric dynamics, we often use the so-called *geocentric* reference frame that is fixed to the rotating Earth. This frame is apparently a noninertial frame (because with respect to a fixed-in-space reference frame it moves with acceleration).

The so-called apparent forces/accelerations to be considered below arise in the geocentric frame in response to its own acceleration.

2. Centripetal acceleration

Imagine a body (e.g., a ball) of mass m attached to a string. We whirl this ball through a circle of radius r at a constant angular velocity $\omega = d\theta/dt$ = [rad s^{-1}]. For the external observer placed in the inertial frame, the ball keeps the same velocity magnitude but the direction of the velocity vector **V** permanently changes, see Fig. 1.4 in the textbook, so the ball perpetually experiences an acceleration.

Consider the change of velocity $\delta\mathbf{V}$ that occurs over the time increment δt, during which the ball rotates through an angle $\delta\theta$ (it may be shown – please do it yourself – that this angle is equal to the angle between directions of vector **V** at times t and $t+\delta t$).

Taking into account that for small $\delta\theta$ one may write $|\delta\mathbf{V}| = |\mathbf{V}|\,\delta\theta$, and tending δt to zero we come to the expression of the time rate of velocity change due to the force (in this case, imposed by the pulling string) that constantly provides acceleration to the ball directed toward the axis of rotation:

$$\frac{d\mathbf{V}}{dt} = |\mathbf{V}|\frac{d\theta}{dt}\left(-\frac{\mathbf{r}}{r}\right) = -|\mathbf{V}|\frac{d\theta}{dt}\hat{\mathbf{r}} = -|\mathbf{V}|\omega\hat{\mathbf{r}},$$

where

$$\frac{d\mathbf{V}}{dt} \equiv \lim_{\delta t\to 0}\frac{\delta\mathbf{V}}{\delta t} = \lim_{\delta t\to 0}\frac{\mathbf{V}+\delta\mathbf{V}-\mathbf{V}}{\delta t} = \lim_{\delta t\to 0}\frac{\mathbf{V}(t+\delta t)-\mathbf{V}(t)}{\delta t},$$

(see Class 13), is the so-called *total derivative* of **V** over time (denoted here as d/dt; note that Holton and Hakim denote the total derivative also as D/Dt; we will discuss such derivative type and its notation in detail later), r

$= |\mathbf{r}|$, $\omega = d\theta/dt$ is the angular velocity of the ball rotation, and signs are chosen according to the directions of vectors $\delta\mathbf{V}$ and $\hat{\mathbf{r}}$.

The considered acceleration $d\mathbf{V}/dt$ is directed toward the center of rotation as one can see from the plot. So, it would be natural to call it the *centripetal acceleration*. The associated force $m(d\mathbf{V}/dt)$, applied to the ball, is the *centripetal force*.

Recall now that in polar coordinates (**note** that the discussed motion case perfectly suits the polar coordinate treatment), the velocity vector under consideration,

$$\mathbf{V} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},$$

does not have a component along $\hat{\mathbf{r}}$, so its magnitude is just given by

$$|\mathbf{V}| = r\frac{d\theta}{dt} = \omega r .$$

Note that velocity $\mathbf{V}$, in general terms, would be expressed as the vector product of the angular velocity vector $\boldsymbol{\omega}$ and the position vector $\mathbf{r}$. In the considered motion case, however, vectors $\mathbf{r}$ and $\boldsymbol{\omega}$ are normal to each other, so their vector product reduces to $\omega r\hat{\boldsymbol{\theta}}$.

Now we can substitute $|\mathbf{V}| = \omega r$ into $d\mathbf{V}/dt = -|\mathbf{V}|\omega\hat{\mathbf{r}}$ (obtained above) to get for the centripetal acceleration of the whirling ball the following expression:

$$\frac{d\mathbf{V}}{dt} = -\omega^2\mathbf{r} .$$

Note that this expression for centripetal acceleration is consistent with the general expression for acceleration in 2-D polar coordinates obtained in Class 16:

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}},$$

which reduces to

$$\mathbf{a} = r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}} = \left(\frac{d\theta}{dt}\right)^2\mathbf{r}$$

in the considered case of motion.

3. Centrifugal force

In the coordinate system (reference frame) rotating with the ball, the ball experiences acceleration (which would be a force divided by mass *m* of the ball) from a string that pulls the ball to the center of rotation. But the ball motion is steady in this reference frame, so the total acceleration of the ball in this reference frame should be zero. To provide for this motion steadiness, there should be a balancing force present in the system that is directed opposite to the pulling force of the string. This balancing force is called the *centrifugal* force.

4. Gravity force

Let us apply now concepts of centripetal and centrifugal forces to an air parcel in the Earth's atmosphere. Such an air parcel, which is at rest with respect to the surface of the Earth (this means that it is placed in the rotating – noninertial – frame attached to the Earth), is thus (see p. **3**) subject to a centrifugal force (acceleration)

$$\mathbf{F}_{Cf} = \Omega^2 \mathbf{R} \,,$$

where $\Omega = 2\pi / 86400 = 7.29 \cdot 10^{-5}$ rad s^{-1} is (the magnitude of) the angular velocity of the Earth's rotation, and **R** is the position vector from the axis of rotation to the parcel.

Considering the joint effect of gravitational force (acceleration) $\mathbf{g}*$ (see Class 19) and centrifugal force at the location of the body as shown in Fig. 1.5 of the textbook, we introduce the *gravity acceleration*

$$\mathbf{g} \equiv \mathbf{g}^* + \Omega^2 \mathbf{R} \,,$$

which is a sum of vectors $\mathbf{g}*$ and $\Omega^2\mathbf{R}$ (and thus it is a vector itself), and the associated *gravity force* $\mathbf{F}_g = m\mathbf{g}$. The above consideration means, in particular, that the gravity force is not directed exactly toward the center of the Earth except at the poles and equator.

5. Geopotential

One can introduce a potential function Φ (called the *geopotential*), whose gradient equals − **g**, i.e.,

$$\nabla\Phi = \mathbf{i}\frac{\partial\Phi}{\partial x} + \mathbf{j}\frac{\partial\Phi}{\partial y} + \mathbf{k}\frac{\partial\Phi}{\partial z} = -\mathbf{g} \,.$$

Taking into account that the vertical coordinate z in atmospheric applications is usually set in the way that

$$\mathbf{g} = 0\mathbf{i} + 0\mathbf{j} - g\mathbf{k} = -g\mathbf{k} \,,$$

where $g \equiv |\mathbf{g}|$, and horizontal changes of the geopotential are typically much smaller than its vertical changes, Φ may approximately be considered as a function of z only, $\Phi = \Phi(z)$, and therefore

$$d\Phi / dz = g \,.$$

If we set $\Phi = 0$ at $z = 0$ (commonly taken at mean sea level), then

$$\Phi(z) = \int_0^z g\,dz$$

corresponds to the amount of work needed to raise a body of unit mass from level $z = 0$ to z.