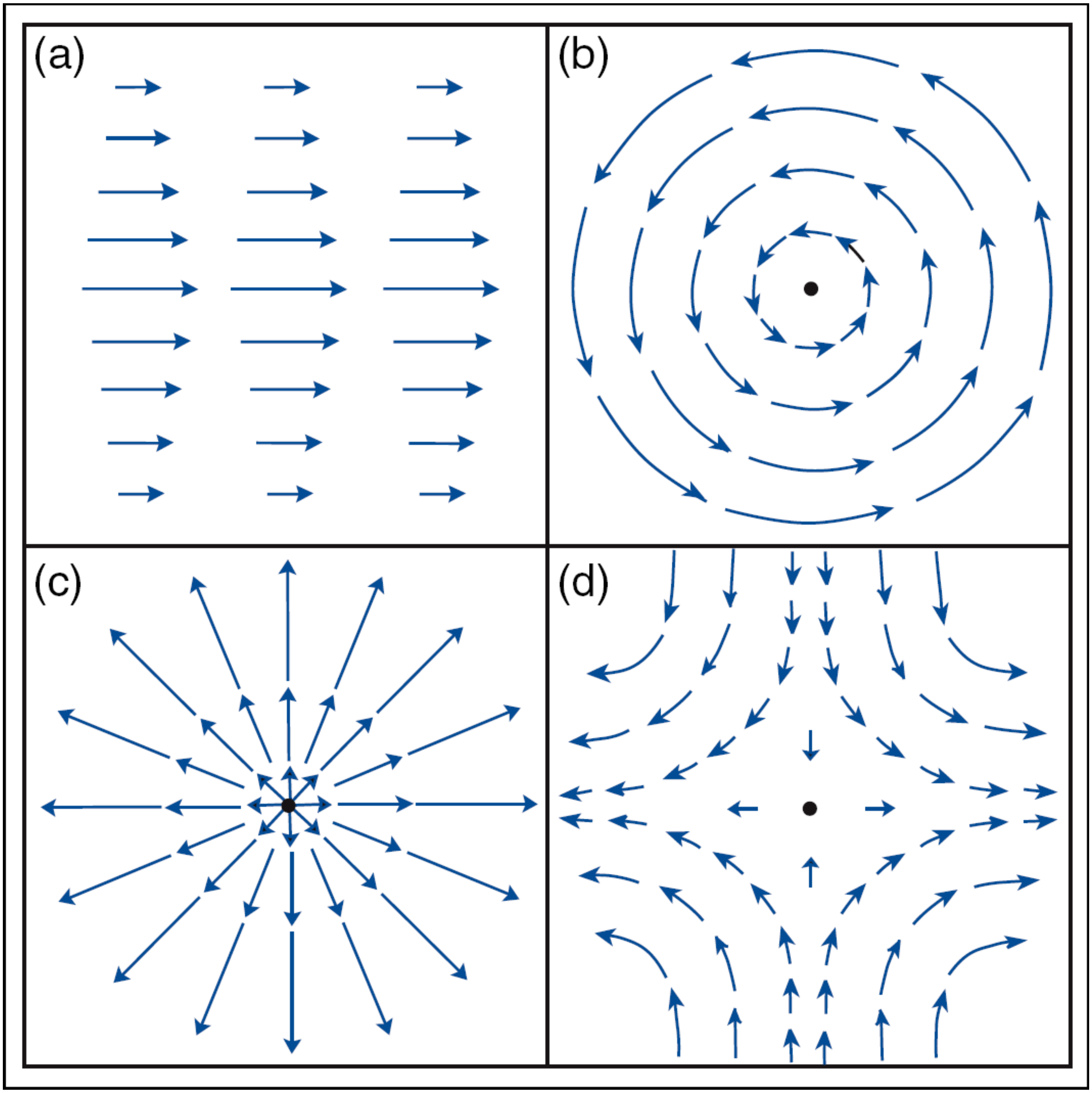
Lecture 17. October 3, 2016

Topics: Examples of differential vector operations with 2-D velocity fields. Differential vector operations with 2-D velocity fields in polar coordinates.

Reading: Appendix C of Holton and Hakim, sections 3 and 10 of Fiedler.

1. Examples of differential vector operations with 2-D velocity fields

We consider four examples of vector velocity field **V** schematically shown in the plot below borrowed from the textbook of Wallace and Hobbs.



In all considered velocity patterns, X is a horizontal coordinate axis directed to the right, and Y is a vertical axis directed upwards.

a. Flow field with x-component of velocity periodically varying in y (so u does not change with x): $u(x,y) = V_0 \cos(2\pi y / L_0)$ and zero y-component of velocity (v=0), where parameter $V_0 > 0$ has dimension of velocity (expressed, e.g., in m s^{-1}) and parameter $L_0 > 0$ has dimension of length (expressed, e.g., in m). Parameter V_0 may be considered as velocity scale of the flow in **a.**, and parameter L_0 may be considered as length scale of this flow.

b. Flow field with x-component of velocity changing with y as $u(x,y) = -V_0 y / L_0$ and y-component of velocity changing with x as $v(x,y) = V_0 x / L_0$, with $V_0 > 0$ and $L_0 > 0$.

c. Flow field with x-component of velocity changing with x as $u(x,y) = V_0 x / L_0$ and y-component of velocity changing with y as $v(x,y) = V_0 y / L_0$, with $V_0 > 0$ and $L_0 > 0$.

d. Flow field with x-component of velocity changing with x as $u(x,y) = V_0 x / L_0$ and y-component of velocity changing with y as $v(x,y) = -V_0 y / L_0$, with $V_0 > 0$ and $L_0 > 0$.

We now evaluate divergence and vorticity of these fields.

a. $u = V_0 \cos(2\pi y / L_0)$, v=0,

Divergence: $\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ everywhere.

Vorticity: $\mathbf{k} \cdot (\nabla \times \mathbf{V}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\pi \frac{V_0}{L_0} \sin(2\pi y / L_0)$. It is zero at y=0 and $2\pi \frac{V_0}{L_0} \sin(2\pi \frac{L_0}{4} / L_0) = 2\pi \frac{V_0}{L_0}$ at $y = \frac{L_0}{4}$ (for example; other y values may also be considered).

b. $u = -V_0 y / L_0$, $v = V_0 x / L_0$,

Divergence: $\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ everywhere.

Vorticity: $\mathbf{k} \cdot (\nabla \times \mathbf{V}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2V_0 / L_0$ everywhere.

c. $u = V_0 x / L_0$, $v = V_0 y / L_0$,

Divergence: $\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2V_0 / L_0$ everywhere.

Vorticity: $\mathbf{k}\cdot(\nabla\times\mathbf{V})=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$ everywhere.

d. $u=V_0 x/L_0$, $\quad v=-V_0 y/L_0$,

Divergence: $\nabla\cdot\mathbf{V}=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0$ everywhere.

Vorticity: $\mathbf{k}\cdot(\nabla\times\mathbf{V})=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$ everywhere.

2. Differential vector operations with 2-D velocity fields in polar coordinates

In polar coordinates, the velocity vector is represented as

$$\mathbf{V}=\frac{dr}{dt}\hat{\mathbf{r}}+r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},$$

(see Class 16), so the velocity components (they are called *radial* and *tangential*, respectively) are

$$v_r=\frac{dr}{dt} \text{ and } v_\theta=r\frac{d\theta}{dt}.$$

Using

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\arctan(y/x),$$

together with the following relationships between velocity components in Cartesian and polar coordinates (see Class 16):

$$v_r=\frac{dr}{dt}=\frac{xu+yv}{\sqrt{x^2+y^2}} \text{ and } v_\theta=r\frac{d\theta}{dt}=\frac{xv-yu}{\sqrt{x^2+y^2}},$$

we may represent velocity fields, considered in p. **2**, in polar coordinates and also evaluate divergence and vorticity of these fields in polar coordinates.

1a. For the flow type **a** from p. **1**, $u=V_0\cos(2\pi y/L_0)$ and v=0, we have

$$v_r=\frac{xV_0\cos(2\pi y/L_0)}{\sqrt{x^2+y^2}}=V_0\cos\theta\cos(2\pi r\sin\theta/L_0) \text{ and } v_\theta=-V_0\sin\theta\cos(2\pi r\sin\theta/L_0).$$

Problem 1. Evaluate divergence $\nabla\cdot\mathbf{V}=\dfrac{1}{r}\dfrac{\partial rv_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta}$ (see Class 17) for this case and show that it is 0 everywhere.

Problem 2. Evaluate (vertical) vorticity $\mathbf{k}\cdot\nabla\times\mathbf{V}=\dfrac{1}{r}\dfrac{\partial(rv_\theta)}{\partial r}-\dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta}$ (see Class 17) at ($r=L_0/4$, $\theta=\pi/2$) and show that it is equal to $2\pi\dfrac{V_0}{L_0}$.

1b. For the flow type **b** from p. **1**, with $u = -V_0 y / L_0$ and $v = V_0 x / L_0$, the radial and tangential velocity components are

$$v_r = 0 \text{ and } v_\theta = V_0 \frac{\sqrt{x^2 + y^2}}{L_0} = \frac{V_0}{L_0} r \,.$$

Those radial and tangential velocity components nicely represent the flow pattern that is composed of concentric circles.

Problem 3. Show that in this case divergence $\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$ everywhere and (vertical) vorticity $\mathbf{k} \cdot \nabla \times \mathbf{V} = \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$ is $2V_0 / L_0$ everywhere.

1c. For the flow type **c** from p. **1**, with $u = V_0 x / L_0$ and $v = V_0 y / L_0$, the polar velocity components are

$$v_r = \frac{V_0}{L_0} r \text{ and } v_\theta = 0 \,.$$

Problem 4. Show that in this case divergence $\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 2V_0 / L_0$ everywhere and (vertical) vorticity $\mathbf{k} \cdot \nabla \times \mathbf{V} = \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0$ everywhere.

1d. For the flow type **d** from p. **1**, with $u = V_0 x / L_0$ and $v = -V_0 y / L_0$, the polar velocity components are

$$v_r = V_0 \frac{r}{L_0} (\cos^2 \theta - \sin^2 \theta) \text{ and } v_\theta = -V_0 \frac{r}{L_0} \sin\theta \cos\theta \,.$$

Problem 5. Show that in this case divergence $\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$ everywhere and (vertical) vorticity $\mathbf{k} \cdot \nabla \times \mathbf{V} = \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0$ everywhere.