Lecture 16. September 30, 2016

Topics: Summary of differential vector operations in Cartesian coordinates on a plane. Polar-coordinate forms of differential vector operators. Relation between directional differential and gradient in polar coordinates.

Reading: Appendix C of Holton and Hakim, sections 3 and 10 of Fiedler.

1. Summary of differential vector operations in Cartesian coordinates on a plane

Of all vector differential operators discussed in Classes 13 to 15, we will focus on the following ones considered in 2-D Cartesian coordinates, that is on the (*X*, *Y*) plane.

1. Gradient of a 2-D scalar field $p = p(x, y)$: $\nabla p = \mathbf{i}\dfrac{\partial p}{\partial x} + \mathbf{j}\dfrac{\partial p}{\partial y}$ (we may conventionally assume *p* to be the atmospheric pressure).

2. Divergence of a 2-D vector field $\mathbf{V} = \mathbf{V}(x, y)$: $\nabla \cdot \mathbf{V} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} \equiv \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ (we may conventionally assume $\mathbf{V}$ to be a horizontal wind velocity vector).

3. Curl of 2-D vector field $\mathbf{V} = \mathbf{V}(x, y)$: $\nabla \times \mathbf{V} = \mathbf{k}\left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right) \equiv \mathbf{k}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)$. It is often presented in coordinate-projection form as $\mathbf{k} \cdot (\nabla \times \mathbf{V}) = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$. If $\mathbf{V}$ is a horizontal velocity vector, then $\zeta \equiv \omega_z = \mathbf{k} \cdot \boldsymbol{\omega} = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ is the vertical component of *vorticity* vector $\boldsymbol{\omega} = \nabla \times \mathbf{V}$ or, simply, *vertical vorticity*.

4. Laplacian (Laplace operator) of a 2-D scalar field $p = p(x, y)$: $\nabla^2 p \equiv \nabla \cdot \nabla p = \dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial y^2} \equiv \Delta p$ (we may again conventionally assume *p* to be pressure).

2. Polar-coordinate forms of vector operations

1. Polar-coordinate form of the *p* gradient, ∇p, was obtained in Class 16:

$$\nabla p = \frac{\partial p}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\boldsymbol{\theta}}.$$

2. Divergence in polar coordinates is given by

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta},$$

where v_r and v_θ are, respectively, r and θ components of $\mathbf{V}$, with x, y, and r, θ related through $x = r\cos\theta$ and $y = r\sin\theta$.

To obtain this expression, use

$$v_r = u\cos\theta + v\sin\theta,$$

$$v_\theta = -u\sin\theta + v\cos\theta,$$

to get

$$u = v_r\cos\theta - v_\theta\sin\theta,$$

$$v = v_r\sin\theta + v_\theta\cos\theta.$$

Differentiate u partially with respect to x to get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}(v_r\cos\theta - v_\theta\sin\theta) + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}(v_r\cos\theta - v_\theta\sin\theta)$$

$$= \frac{\partial r}{\partial x}\left(\frac{\partial v_r}{\partial r}\cos\theta - \frac{\partial v_\theta}{\partial r}\sin\theta\right) + \frac{\partial \theta}{\partial x}\left(\frac{\partial v_r}{\partial \theta}\cos\theta - v_r\sin\theta - \frac{\partial v_\theta}{\partial \theta}\sin\theta - v_\theta\cos\theta\right),$$

and use

$$x = r\cos\theta,\ \ y = r\sin\theta,$$

together with

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta,\ \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\arctan\frac{y}{x} = -\frac{1}{1+\frac{y^2}{x^2}}\frac{y}{x^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r},$$

to obtain

$$\frac{\partial u}{\partial x} = \left(\frac{\partial v_r}{\partial r}\cos\theta - \frac{\partial v_\theta}{\partial r}\sin\theta\right)\cos\theta - \left(\frac{\partial v_r}{\partial \theta}\cos\theta - v_r\sin\theta - \frac{\partial v_\theta}{\partial \theta}\sin\theta - v_\theta\cos\theta\right)\frac{\sin\theta}{r}.$$

Differentiate v partially with respect to y,

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r}(v_r\sin\theta + v_\theta\cos\theta) + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}(v_r\sin\theta + v_\theta\cos\theta)$$

$$= \frac{\partial r}{\partial y}\left(\frac{\partial v_r}{\partial r}\sin\theta + \frac{\partial v_\theta}{\partial r}\cos\theta\right) + \frac{\partial \theta}{\partial y}\left(\frac{\partial v_r}{\partial \theta}\sin\theta + v_r\cos\theta + \frac{\partial v_\theta}{\partial \theta}\cos\theta - v_\theta\sin\theta\right),$$

and use

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta,\ \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\arctan\frac{y}{x} = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r},$$

to obtain

$$\frac{\partial v}{\partial y} = \left(\frac{\partial v_r}{\partial r}\sin\theta + \frac{\partial v_\theta}{\partial r}\cos\theta\right)\sin\theta + \left(\frac{\partial v_r}{\partial \theta}\sin\theta + v_r\cos\theta + \frac{\partial v_\theta}{\partial \theta}\cos\theta - v_\theta\sin\theta\right)\frac{\cos\theta}{r}.$$

Finally, bring $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ together to get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\nabla\cdot\mathbf{V}=\frac{v_r}{r}+\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=\frac{1}{r}\frac{\partial r v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}.$$

3. Vertical (z) component of the curl of a 2-D horizontal vector field $\mathbf{V}$ is given in cylindrical coordinates ($r,\ \theta,\ z$) by

$$\mathbf{k}\cdot(\nabla\times\mathbf{V})=\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta},$$

where v_r and v_θ are, respectively, r and θ components of $\mathbf{V}$.

To derive this expression, write vertical component of the curl in 2-D Cartesian coordinates as (see p. **1**.***3***):

$$\mathbf{k}\cdot\nabla\times\mathbf{V}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

and proceed using

$$u=v_r\cos\theta-v_\theta\sin\theta,\ v=v_r\sin\theta+v_\theta\cos\theta,$$

$$\frac{\partial r}{\partial x}=\cos\theta,\ \frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{r},\ \frac{\partial r}{\partial y}=\sin\theta,\ \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r},$$

analogously to p. **2**.***2*** above (case of divergence) to express $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ through v_r, v_θ, r, and θ. Then subtract $\frac{\partial u}{\partial y}$ from $\frac{\partial v}{\partial x}$ to get

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}.$$

4. Laplace operator (Laplacian) of a differentiable scalar field φ in polar coordinates on a plane is given by

$$\nabla^2\varphi\equiv\nabla\cdot\nabla\varphi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}.$$

To obtain this expression, start from its 2-D Cartesian counterpart:

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}.$$

Differentiate φ with respect to x to get

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \varphi}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

Then differentiate the obtained expression to get

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial \varphi}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 \varphi}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial \varphi}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + 2\frac{\partial^2 \varphi}{\partial r \partial \theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}.$$

Relating polar coordinates r and θ to the plane Cartesian coordinates x and y through

$$x = r\cos\theta,\ y = r\sin\theta,\ r = \sqrt{x^2 + y^2},\ \theta = \arctan(y/x),$$

we find

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta,\ \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r},\ \frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{\sin^2\theta}{r},$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} = \frac{2\sin\theta\cos\theta}{r^2},$$

and arrive at

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial r^2}\cos^2\theta + \frac{\partial \varphi}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial^2 \varphi}{\partial \theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial \varphi}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} - 2\frac{\partial^2 \varphi}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}.$$

Analogously, one can obtain the following expression for the second derivative of φ with respect to y:

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial r^2}\sin^2\theta + \frac{\partial \varphi}{\partial r}\frac{\cos^2\theta}{r} + \frac{\partial^2 \varphi}{\partial \theta^2}\frac{\cos^2\theta}{r^2} - \frac{\partial \varphi}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} + 2\frac{\partial^2 \varphi}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}.$$

Summing up the above expressions, we come up with

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2\varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}.$$

3. Relation between directional differential and gradient in polar coordinates

In Class 13 we considered a displacement vector $d\mathbf{m}$ in Cartesian space and obtained the following expression for the directional differential dT of the scalar variable T due to the displacement $d\mathbf{m}$ (we look here at the 2-D version of that expression):

$$dT = \left(\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}\right)\cdot\left(dx\mathbf{i} + dy\mathbf{j}\right) = \nabla T \cdot d\mathbf{m} = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy,$$

where $d\mathbf{m} = dx\mathbf{i} + dy\mathbf{j}$ is the displacement vector expressed in the Cartesian coordinate form.

Now consider the displacement vector $d\mathbf{m}$ in 2-D polar coordinates (r, θ) related to Cartesian coordinates through

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

In the (r, θ) coordinates, the decomposition of $d\mathbf{m}$ is presented by

$$d\mathbf{m} = d\mathbf{r} = d(r\hat{\mathbf{r}}) = dr\hat{\mathbf{r}} + rd\hat{\mathbf{r}} = dr\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}d\theta = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}},$$

where we used $\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}}$, see Class 16.

Assuming that expression for differential $dT = \nabla T \cdot d\mathbf{m}$ holds in polar coordinates, we may write

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta = \nabla T \cdot d\mathbf{m} = (\nabla T_r \hat{\mathbf{r}} + \nabla T_\theta \hat{\boldsymbol{\theta}}) \cdot (dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}}),$$

or, using rules of evaluation of the dot product,

$$\frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta = \nabla T_r dr + \nabla T_\theta r d\theta,$$

from which we conclude that

$$\frac{\partial T}{\partial r} = \nabla T_r \text{ and } \frac{\partial T}{\partial \theta} = r\nabla T_\theta .$$

Therefore, components ∇T_r and ∇T_θ of the gradient ∇T in polar coordinates are

$$\nabla T_r = \frac{\partial T}{\partial r} \text{ and } \nabla T_\theta = \frac{1}{r}\frac{\partial T}{\partial \theta},$$

and the gradient may be written as

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} .$$

This is the same expression that was already obtained, albeit in a different way, in Class16: