Lecture 15. September 28, 2016

Topics: Position vector and unit vectors in polar coordinates. Relationships between velocity components in Cartesian and polar coordinates. Acceleration in 2-D polar coordinates. Gradient in 2-D polar coordinates.

Reading: Appendix C of Holton and Hakim, sections 3 and 10 of Fiedler.

1. Position vector and unit vectors in polar coordinates

Polar coordinate system was considered in Class 5. It was shown that polar coordinates r and θ are converted to the plane (2-D) Cartesian coordinates x and y as

$$x = r\cos\theta \text{ and } y = r\sin\theta .$$

Reversely, 2-D Cartesian coordinates x and y can be converted to polar coordinates r and θ with $r \geq 0$ and θ in the interval $(-\pi, \pi]$ as

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan(y / x) .$$

Coordinates $x = r\cos\theta$ and $y = r\sin\theta$ may be considered as coordinates of the position vector $\mathbf{r}$ (Class 4),

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} ,$$

which may also be in terms of the radius r and directional angle θ. By writing

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j},$$

one may introduce a unit vector $\hat{\mathbf{r}}$ as

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} .$$

Note the analogy to the way a unit vector $\hat{\mathbf{b}}$ of an arbitrary vector $\mathbf{b}$ was introduced in Class 9.

In order to fully characterize vector $\mathbf{r}$, we also need to specify the direction of change of the second polar coordinate, which is angle θ. The unit vector in the θ direction, $\hat{\boldsymbol{\theta}}$, has a unit magnitude and is directed orthogonally to $\mathbf{r}$, so that coordinate basis ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$) obeys the right-hand convention (see the plot below). Therefore, with respect to the companion 2-D Cartesian coordinate system, the polar coordinate directions $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are presented by

$$\hat{\mathbf{r}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta = \hat{\mathbf{r}}(\theta) ,$$

$$\hat{\boldsymbol{\theta}} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta = \hat{\boldsymbol{\theta}}(\theta) ,$$

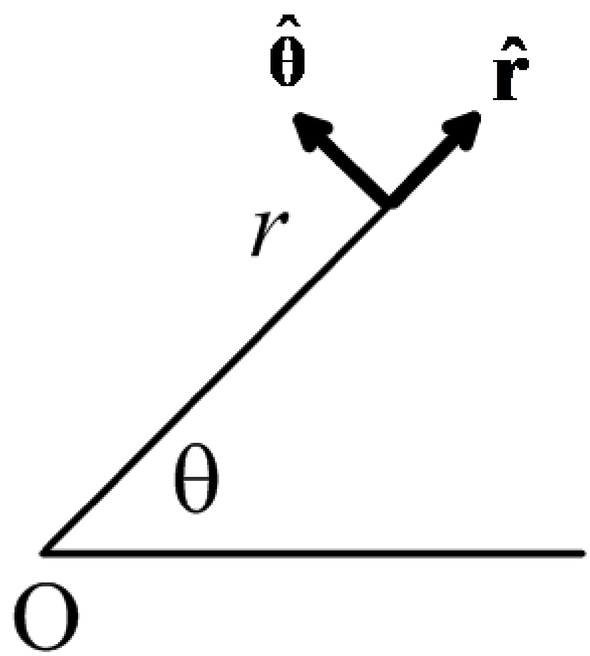
which satisfy the condition $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$.

Note that the above expressions would also describe components of a basis in the Cartesian system rotated by angle θ with respect to the original system with $\mathbf{i}$, $\mathbf{j}$ basis.

Different to Cartesian coordinates, where the same basis is applied for all positions and thus x and y directions are independent, so that $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$ and $\frac{\partial \mathbf{i}}{\partial y} = \frac{\partial \mathbf{j}}{\partial x} = 0$, the fundamental directions in polar coordinates are changing in relation to each other (because $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ both depend on θ), which provides

$$\frac{d\hat{\mathbf{r}}}{d\theta} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta = \hat{\boldsymbol{\theta}},$$

$$\frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\mathbf{i}\cos\theta - \mathbf{j}\sin\theta = -\hat{\mathbf{r}} .$$



2. Relationships between velocity components in Cartesian and polar coordinates

In polar coordinates, velocity vector $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is therefore given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr\hat{\mathbf{r}}(\theta)}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}(\theta)}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}},$$

where we used (see p. **1**):

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}} \text{ and } \frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}.$$

Thus, the polar-system velocity components (they are called *radial* and *tangential* velocity components, respectively) are

$$v_r = \frac{dr}{dt} \text{ and } v_\theta = r\frac{d\theta}{dt},$$

and the velocity vector may be presented as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} .$$

Using

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan(y/x) ,$$

one can obtain the following relationships between velocity components in Cartesian and polar coordinates (please obtain those yourself):

$$v_r = \frac{dr}{dt} = \frac{xu + yv}{\sqrt{x^2 + y^2}} \text{ and } v_\theta = r\frac{d\theta}{dt} = \frac{xv - yu}{\sqrt{x^2 + y^2}} .$$

3. Acceleration in 2-D polar coordinates

Acceleration $\mathbf{a}$ is given by a total time derivative of velocity. Thus (see p. **2**):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{dr}{dt}\hat{\mathbf{r}} + \frac{d}{dt}r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

$$= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} ,$$

where we used

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} , \; \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}} \text{ and } \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} .$$

Therefore,

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} ,$$

or

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} ,$$

where

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 , \; a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} .$$

Another way to obtain the above expression for acceleration in polar coordinates would be to differentiate

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

as

$$\mathbf{a} = \frac{d}{dt}(v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}) ,$$

using

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = \frac{v_\theta}{r}\hat{\boldsymbol{\theta}} \text{ and } \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} = -\frac{v_\theta}{r}\hat{\mathbf{r}} .$$

Please do this derivation yourself.

4. Gradient in 2-D polar coordinates

Unit vectors in the polar system and 2-D Cartesian system are related as

$$\hat{\mathbf{r}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta\,,$$

$$\hat{\boldsymbol{\theta}} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta\,.$$

Expressing them inversely, we have

$$\mathbf{i} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\,,$$

$$\mathbf{j} = \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta\,.$$

Now express

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j},$$

where φ is an arbitrary differentiable scalar function, as

$$\nabla\varphi = (\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\frac{\partial\varphi}{\partial x} + (\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\frac{\partial\varphi}{\partial y}$$
$$= (\cos\theta\frac{\partial\varphi}{\partial x} + \sin\theta\frac{\partial\varphi}{\partial y})\hat{\mathbf{r}} + (-\sin\theta\frac{\partial\varphi}{\partial x} + \cos\theta\frac{\partial\varphi}{\partial y})\hat{\boldsymbol{\theta}},$$

and use (see Class 5):

$$\frac{\partial\varphi}{\partial r} = \frac{x}{r}\frac{\partial\varphi}{\partial x} + \frac{y}{r}\frac{\partial\varphi}{\partial y} \text{ and } \frac{\partial\varphi}{\partial\theta} = -y\frac{\partial\varphi}{\partial x} + x\frac{\partial\varphi}{\partial y}\,,$$

to obtain

$$\frac{\partial\varphi}{\partial x} = \frac{x}{r}\frac{\partial\varphi}{\partial r} - \frac{y}{r^2}\frac{\partial\varphi}{\partial\theta} \text{ and } \frac{\partial\varphi}{\partial y} = \frac{y}{r}\frac{\partial\varphi}{\partial r} + \frac{x}{r^2}\frac{\partial\varphi}{\partial\theta}\,.$$

Taking into account that $x = r\cos\theta$ and $y = r\sin\theta$, this provides

$$\frac{\partial\varphi}{\partial x} = \cos\theta\frac{\partial\varphi}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial\varphi}{\partial\theta} \text{ and } \frac{\partial\varphi}{\partial y} = \sin\theta\frac{\partial\varphi}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial\varphi}{\partial\theta}\,.$$

Substituting the above relationships in the expression for $\nabla\varphi$, we come to

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\boldsymbol{\theta}}\,.$$