**Lecture 14.** September 26, 2016

**Topics:** Laplace operator (Laplacian). Divergence theorem of vector calculus. Divergence theorem and continuity equation; mass flux.

**Reading:** Appendix C and section 2.5 in Holton and Hakim.

## 1. Laplace operator (Laplacian)

The *Laplace operator* (also called *Laplacian*), commonly denoted as $\nabla^2$ (often also denoted as $\nabla\cdot\nabla$ or $\Delta$) is a second order differential operator in the *n*-dimensional Euclidean space, defined with respect to a scalar function $f = f(x_i,\ i=1,\ \ldots,\ n)$, as the divergence ($\nabla\cdot$) of $\nabla f$, the gradient of $f$. Thus,

$$\nabla^2 f = \nabla\cdot\nabla f\ .$$

In 3-D space ($n$=3) with Cartesian coordinates $x = x_1,\ y = x_2,\ z = x_3$, the Laplacian appears as (see properties of gradient and divergence operators considered in Class 12):

$$\nabla^2 f = \nabla\cdot\nabla f = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})\cdot(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \Delta f\ .$$

In 2-D Cartesian coordinate system on a plane ($x$, $y$), when $f = f(x, y)$ the above expression reduces to

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f\ .$$

Sometimes in this case, the subscript $h$ is added to the Laplacian operator to indicate that it applies only in horizontal directions, i.e. it is written as $\Delta_h f \equiv \nabla_h{}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

In general form, the Laplacian is expressed through the sum of partial derivatives of $f$ as

$$\nabla^2 f = \nabla\cdot\nabla f = \sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i{}^2} = \Delta f\ .$$

In some atmospheric dynamics considerations, the Laplacian is also applied to vector fields. For instance, the result of $\nabla^2$ being applied to a vector field $\mathbf{V} = \mathbf{V}(x, y, z) = (u, v, w)$ would be a vector

$$\nabla^2\mathbf{V} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\mathbf{k}\,.$$

## 2. Divergence theorem of vector calculus

In atmospheric dynamics, we commonly employ a 3-D version of this prominent mathematical theorem.

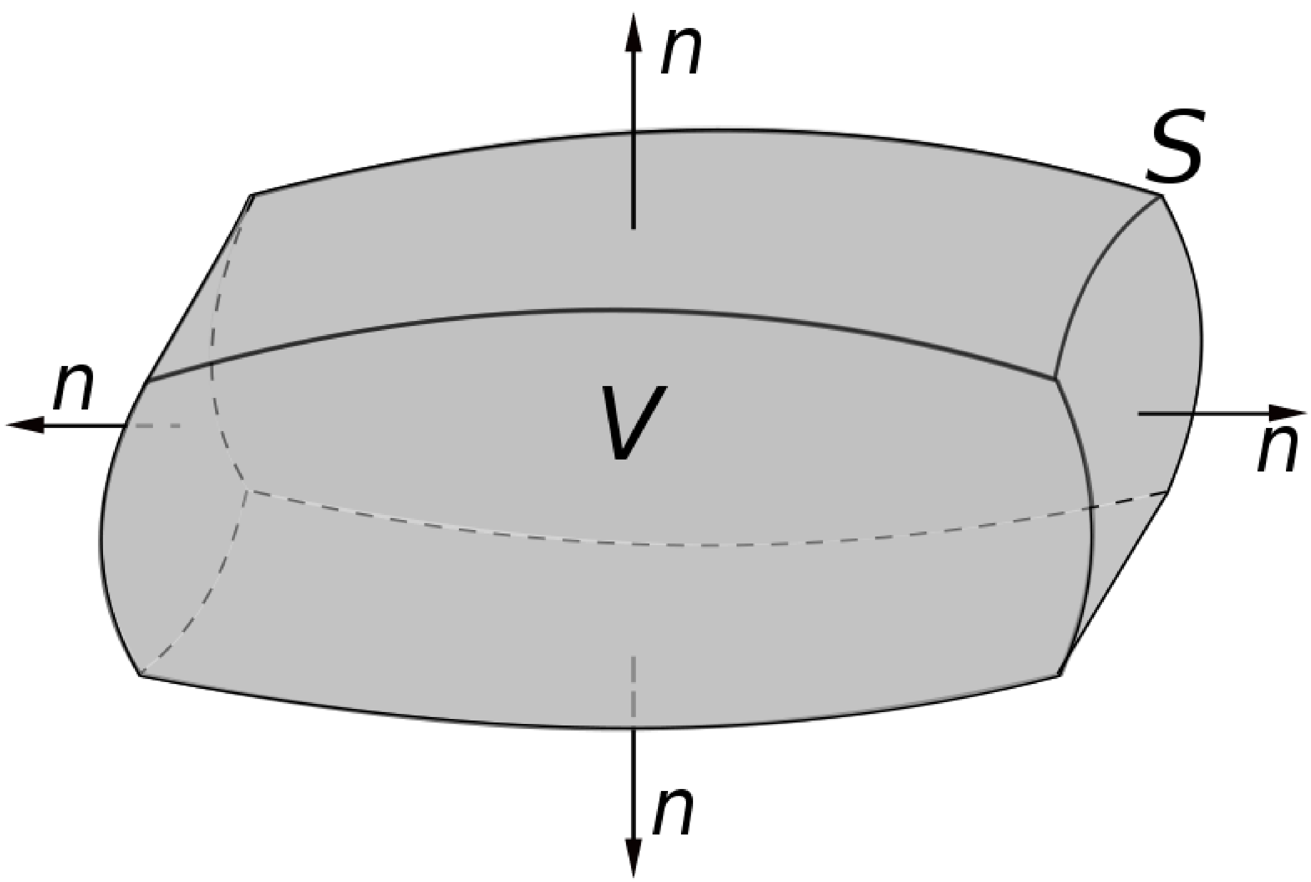
Consider a (compact) volume $V$ in 3-D space which has a (piecewise smooth) boundary $A$. If $\mathbf{B}$ is a continuously differentiable vector field defined within $V$, then we have

$$\int_V \nabla \cdot \mathbf{B}\, dV = \int_A \mathbf{B} \cdot \mathbf{n}\, dA\,.$$

This is the formal statement of the so-called *divergence theorem of vector calculus*. On the left-hand side is an integral over the volume *V*. On the right side is the surface integral over the boundary of the volume *V*, where **n** is the outward pointing local unit vector normal to the boundary. Often you may find $\mathbf{n}\, dA$ on the right-hand side denoted as $d\mathbf{A} = \mathbf{n}\, dA$, so that the statement of the theorem appears as

$$\int_V \nabla \cdot \mathbf{B}\, dV = \int_A \mathbf{B} \cdot d\mathbf{A}\,.$$

Qualitatively, the above expression represents the balance between the integrated over the volume *V* divergence of vector field **B** and a surface integral of vector **B** over the surface *A* (denoted as *S* in an illustrative plot below) that surrounds volume *V*.




**3. Divergence theorem and continuity equation; mass flux**

If we take *V* as a fixed volume in the moving fluid and **B** as a vector of momentum per unit volume, i.e. $\mathbf{B} = \rho \mathbf{U}$, where $\mathbf{U}$ is the fluid velocity vector, and $\rho$ is the fluid density, then $\rho \mathbf{U}$ would represent the mass of fluid that passes locally through unit surface area of $A$ per unit time (indeed, the unit of $\rho\mathbf{U}$ is $\mathrm{kg\ m^{-3} m\ s^{-1} = kg\ m^{-2}\ s^{-1}}$), i.e., the *mass flux*.

Correspondingly, the integral mass flux through the surface $A$, $\int_A \rho\mathbf{U}\cdot d\mathbf{A}$, will represent the total mass of fluid that left the volume $V$ (if the integral is positive) or entered the volume $V$ (if the integral is negative) per unit time. This change of mass due to the flux through the boundary should be equal to the total change of mass in the volume:

$$-\frac{d}{dt}\int_V \rho\, dV = -\int_V \frac{\partial\rho}{\partial t}\, dV = \int_A \rho\mathbf{U}\cdot d\mathbf{A},$$

where

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial\rho}{\partial t}\, dV$$

because the volume is fixed, while the minus sign takes into account that the mass of fluid in the volume is decreasing when the fluid leaves the volume.

But recall now that, according to the divergence theorem,

$$\int_A \rho\mathbf{U}\cdot d\mathbf{A} = \int_V \nabla\cdot\rho\mathbf{U}\, dV,$$

which means that integral of the flux divergence over volume $V$ is equal to the integral of the flux over a surface $A$ that surrounds the volume $V$, and thus

$$-\int_V \frac{\partial\rho}{\partial t}\, dV = \int_V \nabla\cdot\rho\mathbf{U}\, dV,$$

or

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{U}\right) dV = 0,$$

which can be written in the differential form as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{U} = 0.$$

In atmospheric dynamics, various continuity equation forms are used depending on scales of considered atmospheric motion.