Lecture 13. September 23, 2016

Topics: Examples of vector calculus operations. Invariance of the vector sum magnitude with respect to the basis rotation. Proving selected vector calculus identities. Importance of the order of operands in vector calculations. Example of the gradient calculation. Fun with del operator.

Reading: Appendix C of Holton and Hakim, and section 3.5 of Fiedler.

1. Invariance of the vector sum magnitude with respect to the basis rotation

Consider two vectors, $\vec{a}$ and $\vec{b}$, in two 2-D Cartesian systems, one with basis $\hat{i}, \hat{j}$ and the second with basis $\hat{I}, \hat{J}$. The second basis is rotated by angle θ with respect to the first basis. Writing $\vec{a}$ and $\vec{b}$ down in the component form, prove that magnitude *c* of the vector sum $\vec{c} = \vec{a} + \vec{b}$ is the same in both systems. In other words, prove that magnitude of the vector $\vec{a} + \vec{b}$ is invariant with respect to the basis rotation.

Denoting the coordinate system with $\hat{i}, \hat{j}$ basis as (*X*, *Y*) and the system with basis $\hat{I}, \hat{J}$ as (*X*', *Y*'), we write:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} = a_{x'}\hat{I} + a_{y'}\hat{J} \text{ and } \vec{b} = b_x\hat{i} + b_y\hat{j} = b_{x'}\hat{I} + b_{y'}\hat{J}.$$

Consider projections of $\hat{I}$, $\hat{J}$ unit vectors on coordinate directions *x* and *y* of the original system. These projections would be *x* and *y* coordinates of the $\hat{I}$, $\hat{J}$ unit vectors in the (*X*, *Y*) system. Based on geometrical considerations:

$$\hat{I}_x = \cos\theta, \quad \hat{I}_y = \sin\theta, \qquad \hat{J}_x = -\sin\theta, \quad \hat{J}_y = \cos\theta,$$

and thus

$$\hat{I} = \cos\theta\hat{i} + \sin\theta\hat{j}, \qquad \hat{J} = -\sin\theta\hat{i} + \cos\theta\hat{j}.$$

Then we express, using rules of calculation of vector projections on coordinates axes,

$$a_{x'} = \hat{I}\cdot\vec{a} = (\cos\theta\hat{i} + \sin\theta\hat{j})\cdot(a_x\hat{i} + a_y\hat{j}) = a_x\cos\theta + a_y\sin\theta,$$

$$b_{x'} = \hat{I}\cdot\vec{b} = (\cos\theta\hat{i} + \sin\theta\hat{j})\cdot(b_x\hat{i} + b_y\hat{j}) = b_x\cos\theta + b_y\sin\theta,$$

and analogously obtain

$$a_{y'} = -a_x\sin\theta + a_y\cos\theta, \qquad b_{y'} = -b_x\sin\theta + b_y\cos\theta.$$

The magnitude of vector $\vec{c} = \vec{a} + \vec{b}$ in (*X*', *Y*') coordinates is given by

$$|\vec{c}| = |\vec{a} + \vec{b}| = \sqrt{(a_{x'} + b_{x'})^2 + (a_{y'} + b_{y'})^2}.$$

Substituting $a_{x'}$, $b_{x'}$, $a_{y'}$, and $b_{y'}$ expressed in terms of a_x, b_x, a_y, b_y, and θ, into $(a_{x'} + b_{x'})^2 + (a_{y'} + b_{y'})^2$ we come to

$$(a_{x'}+b_{x'})^2+(a_{y'}+b_{y'})^2=(a_x+b_x)^2+(a_y+b_y)^2,$$

so $|\vec{c}|$ is invariant with respect to the basis rotation.

2. Proving selected vector calculus identities

2a. Assuming 3-D Cartesian coordinates, prove that

$$\nabla\cdot(\nabla\times\mathbf{U})=0.$$

Use the defining expression for ∇ in Cartesian coordinates,

$$\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k},$$

and write vector $\mathbf{U}$ as

$$\mathbf{U}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k},$$

where *u*, *v*, and *w* are Cartesian components of $\mathbf{U}$. The vector product $\nabla\times\mathbf{U}$ (curl of $\mathbf{U}$) is given by

$$\nabla\times\mathbf{U}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\mathbf{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathbf{k},$$

which can be written as

$$\nabla\times\mathbf{U}=\mathbf{D}=D_x\mathbf{i}+D_y\mathbf{j}+D_z\mathbf{k}$$

with

$$D_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\ D_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\ D_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

This brings us to

$$\nabla\cdot(\nabla\times\mathbf{U})=\nabla\cdot\mathbf{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0.$$

2b. Assuming 3-D Cartesian coordinates, prove that

$$\nabla\times\nabla T=0.$$

The quantity ∇T is a vector given by

$$\nabla T=\frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial z}\mathbf{k}.$$

Introduce $\mathbf{D}\equiv\nabla T$ with $D_x\equiv\frac{\partial T}{\partial x},\ D_y\equiv\frac{\partial T}{\partial y},\ D_z\equiv\frac{\partial T}{\partial z}$, and write

$$\nabla\times\nabla T=\nabla\times\mathbf{D}$$

$$=\left(\frac{\partial D_z}{\partial y}-\frac{\partial D_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial D_x}{\partial z}-\frac{\partial D_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial D_y}{\partial x}-\frac{\partial D_x}{\partial y}\right)\mathbf{k}=\left(\frac{\partial^2 T}{\partial y\partial z}-\frac{\partial^2 T}{\partial z\partial y}\right)\mathbf{i}+\left(\frac{\partial^2 T}{\partial x\partial z}-\frac{\partial^2 T}{\partial z\partial x}\right)\mathbf{j}+\left(\frac{\partial^2 T}{\partial x\partial y}-\frac{\partial^2 T}{\partial y\partial x}\right)\mathbf{k}=0.$$

3. Importance of the order of operands in vector calculations

Consider a 3-D Cartesian vector velocity field $\mathbf{V} = \mathbf{V}(x, y, z) = (u, v, w)$. As was indicated in Class 13, operation $(\mathbf{V} \cdot \nabla)\mathbf{V}$ results in a vector (given that **V** is velocity, it would be the vector of advection of momentum per unit mass). **Note** that $\mathbf{V} \cdot \nabla$ is a dot product of a vector and a del operator, so it is a scalar operator. Let us write $(\mathbf{V} \cdot \nabla)\mathbf{V}$ down in component form:

$$\mathbf{V} \cdot \nabla = (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

Enacting this scalar operator on vector $\mathbf{V}$ produces a vector which can be written as

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right)\mathbf{V} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right)\mathbf{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right)\mathbf{j} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right)\mathbf{k}.$$

Note that taking $\nabla \cdot \mathbf{V}$ (divergence of vector **V**, a scalar quantity) instead of the scalar operator $\mathbf{V} \cdot \nabla$ in the above calculations will change a lot. First, we have

$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

Then, calculating $(\nabla \cdot \mathbf{V})\mathbf{V}$ [that is multiplying vector by a scalar, which is a commutative operation so $(\nabla \cdot \mathbf{V})\mathbf{V} = \mathbf{V}(\nabla \cdot \mathbf{V})$], we come up with

$$(\nabla \cdot \mathbf{V})\mathbf{V} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)\mathbf{V} = u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)\mathbf{i} + v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)\mathbf{j} + w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)\mathbf{k},$$

which is also a vector, but not the same one as $(\mathbf{V} \cdot \nabla)\mathbf{V}$! One may consider other examples of similar kind. For instance, the gradient of a scalar field T in Cartesian coordinates, ∇T, is given by

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}} + \frac{\partial T}{\partial z}\hat{\mathbf{k}},$$

while the operation $T\nabla$ will result in the vector operator of the following kind:

$$T\nabla = T\frac{\partial}{\partial x}\hat{\mathbf{i}} + T\frac{\partial}{\partial y}\hat{\mathbf{j}} + T\frac{\partial}{\partial z}\hat{\mathbf{k}}.$$

Another example, involving the vector (cross) product, is $\nabla \times \mathbf{V}$ (curl of vector $\mathbf{V}$) given by

$$\nabla \times \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k}.$$

Changing the order of operands, i.e. writing it as $\mathbf{V} \times \nabla$ results in the vector operator that has the following appearance in component form,

$$\mathbf{V} \times \nabla = \left(v\frac{\partial}{\partial z} - w\frac{\partial}{\partial y} \right)\mathbf{i} + \left(w\frac{\partial}{\partial x} - u\frac{\partial}{\partial z} \right)\mathbf{j} + \left(u\frac{\partial}{\partial y} - v\frac{\partial}{\partial x} \right)\mathbf{k},$$

which is very different to the component form of $\nabla \times \mathbf{V}$ (see above).

4. Example of the gradient calculation

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Prove that

$$\nabla r = \frac{\mathbf{r}}{r}.$$

Writing ∇r as

$$\nabla r = \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial x}\mathbf{k},$$

we evaluate

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{1}{2}\frac{\frac{\partial x^2}{\partial x}}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}.$$

By analogy:

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Therefore,

$$\nabla r = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} = \frac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\mathbf{r}}{r}.$$

5. Fun with del operator

Write down $\text{grad } lounge$ in different possible ways using the del operator and prove that

$$\text{grad } lounge = lounge\nabla \ln lounge.$$

Follow the sequence of expressions presented below:

$$\text{grad } lounge = \nabla lounge = \mathbf{i}\frac{\partial lounge}{\partial x} + \mathbf{j}\frac{\partial lounge}{\partial y} + \mathbf{k}\frac{\partial lounge}{\partial z} =$$

$$ounge\nabla l + lunge\nabla o + longe\nabla u + louge\nabla n + loune\nabla g + loung\nabla e =$$

$$lounge\left(\frac{\nabla l}{l} + \frac{\nabla o}{o} + \frac{\nabla u}{u} + \frac{\nabla n}{n} + \frac{\nabla g}{g} + \frac{\nabla e}{e}\right) =$$

$$lounge\left(\nabla \ln l + \nabla \ln o + \nabla \ln u + \nabla \ln n + \nabla \ln g + \nabla \ln e\right) =$$

$$lounge\left[\nabla(\ln l + \ln o + \ln u + \ln n + \ln g + \ln e)\right] =$$

$$lounge\nabla \ln lounge.$$