Lecture 12. September 19, 2016

Topics: Vector differentiation with respect to a scalar argument. Directional derivative and gradient of a scalar field. Del (nabla) operator. Definitions and properties of divergence and curl.

Reading: Appendix C of Holton and Hakim, and section 3.5 of Fiedler.

1. Vector differentiation

Derivative $\frac{d\mathbf{v}}{dt}$ of vector function $\mathbf{v}$ (e.g., velocity) with respect to a scalar argument t (e.g., time) is defined as

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}.$$

Properties:

$$\frac{d(\mathbf{v}+\mathbf{w})}{dt} = \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{w}}{dt},$$

$$\frac{dc\mathbf{v}}{dt} = c\frac{d\mathbf{v}}{dt} \text{ (where } c \text{ is a scalar constant),}$$

$$\frac{d(\mathbf{v}\cdot\mathbf{w})}{dt} = \mathbf{v}\cdot\frac{d\mathbf{w}}{dt} + \mathbf{w}\cdot\frac{d\mathbf{v}}{dt},$$

$$\frac{d(\mathbf{v}\times\mathbf{w})}{dt} = \mathbf{v}\times\frac{d\mathbf{w}}{dt} + \mathbf{w}\times\frac{d\mathbf{v}}{dt},$$

$$\frac{d|\mathbf{v}|}{dt} = \frac{1}{|\mathbf{v}|}\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \text{ (it is presumed that } \mathbf{v} \neq 0\text{).}$$

Examples of vector functions and vector-function derivatives:

Velocity (e.g., of a particle)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}},$$

where $\mathbf{r}$ is the position vector.

In polar coordinates on a plane, as will be shown later,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr\hat{\mathbf{r}}(\theta)}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}(\theta)}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}.$$

Acceleration,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d}{dt}\frac{d\mathbf{r}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

2. Directional derivative and gradient of a scalar field; del (nabla) operator

Consider a scalar function $T(x,y,z)$ of independent spatial variables x,y,z (we may consider it to be air temperature). Spatial derivative of *T* with respect to *x*, $\frac{\partial T}{\partial x}$, describes the rate of change of *T* in *x* direction (it would be an eastward change in temperature if the *X* axis is directed to the east). **Note** that while evaluating $\frac{\partial T}{\partial x}$, *y* and *z* values are kept constant. Analogously, $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$ describe the rates of temperature change in *y* and *z* directions, respectively.

How could one evaluate the rate of change of temperature in an arbitrary direction?

Consider a spatial change specified by a vector $d\mathbf{m}$ that has coordinate components dx, dy, and dz:

$$d\mathbf{m} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}.$$

Like with other Cartesian vectors, one may introduce the magnitude of this vector, $dm = |d\mathbf{m}|$ and associated unit vector $\hat{\mathbf{m}} = \frac{d\mathbf{m}}{dm}$ (it is pointing in the same direction as $d\mathbf{m}$). In coordinate form, the unit vector $\hat{\mathbf{m}}$ is therefore presented as

$$\hat{\mathbf{m}} = \frac{dx}{dm}\hat{\mathbf{i}} + \frac{dy}{dm}\hat{\mathbf{j}} + \frac{dz}{dm}\hat{\mathbf{k}}.$$

Now consider the aggregated change of temperature $T(x,y,z)$ in *x*, *y*, and *z* directions associated with spatial displacements dx, dy, and dz:

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz.$$

Referring this temperature change to the spatial increment dm, we obtain

$$\frac{dT}{dm} = \frac{\partial T}{\partial x}\frac{dx}{dm} + \frac{\partial T}{\partial y}\frac{dy}{dm} + \frac{\partial T}{\partial z}\frac{dz}{dm}.$$

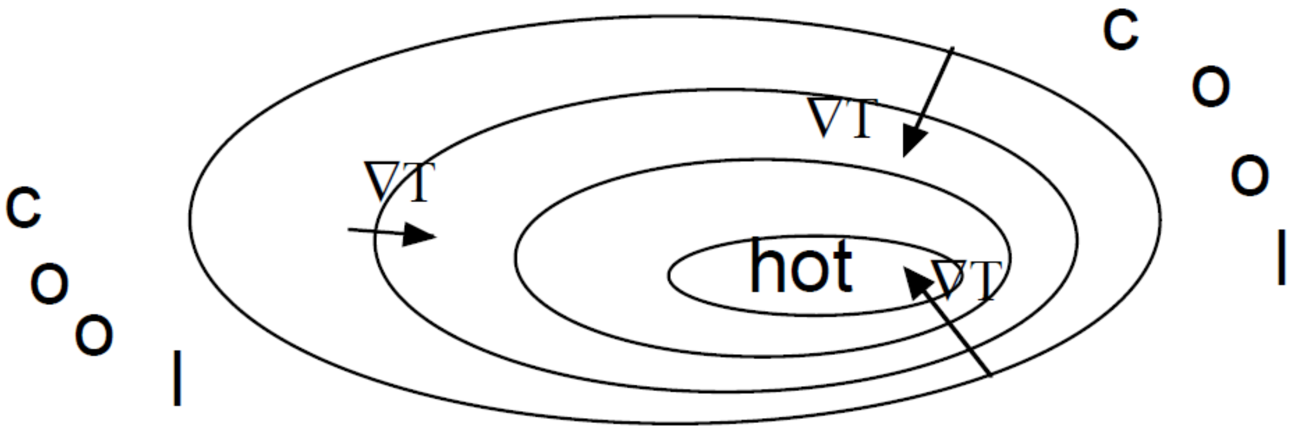
Now recall that $\frac{dx}{dm}, \frac{dy}{dm}, \frac{dz}{dm}$ are components of vector $\hat{\mathbf{m}}$. Introducing vector $\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}} + \frac{\partial T}{\partial z}\hat{\mathbf{k}}$ and employing the dot product formalism, one may therefore write

$$\frac{dT}{dm} = \frac{\partial T}{\partial x}\frac{dx}{dm} + \frac{\partial T}{\partial y}\frac{dy}{dm} + \frac{\partial T}{\partial z}\frac{dz}{dm} = \left(\frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}} + \frac{\partial T}{\partial z}\hat{\mathbf{k}}\right)\cdot\left(\frac{dx}{dm}\hat{\mathbf{i}} + \frac{dy}{dm}\hat{\mathbf{j}} + \frac{dz}{dm}\hat{\mathbf{k}}\right) = \nabla T \cdot \hat{\mathbf{m}},$$

so the directional derivative (rate of change) of *T* in the direction of interest specified by unit vector $\hat{\mathbf{m}}$ is given by a scalar product of the vector ∇T, which is called the *gradient* of scalar field $T(x,y,z)$ (temperature), and the unit vector $\hat{\mathbf{m}}$:

$$\frac{dT}{dm} = \nabla T \cdot \hat{\mathbf{m}}.$$

It follows from the above expression that when the gradient vector is directed normal to $\hat{\mathbf{m}}$, the rate of change $\frac{dT}{dm} = 0$, so ∇T is directed perpendicular to the line (surface) of constant T (in this case it would be called isotherm). When $\hat{\mathbf{m}}$ is in direction of ∇T, the largest positive value of $\frac{dT}{dm}$ is attained, so ∇T points in the direction of greatest spatial change of T from lower to higher temperature values (see illustration below).



3. Del (nabla) operator

Operator ∇, defined as

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}},$$

is called the *del* (or *nabla*) *operator*. It is considered a vector operator because in many respects it behaves like a vector.

In Cartesian coordinates, del operator ∇ acting on an arbitrary scalar field $F(x, y, z)$ produces a vector ∇F with components $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ (as was shown in p. **2** using example of temperature), so

$$\nabla F = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}.$$

Action of the del operator ∇ on an arbitrary vector field $\mathbf{U}(x, y, z)$ has two forms.

Action in the form $\nabla \cdot \mathbf{U}$ (scalar/dot product of ∇ and $\mathbf{U}$) produces a scalar which is called the *divergence* of $\mathbf{U}$. Action in the form $\nabla \times \mathbf{U}$ (vector/cross product of ∇ and $\mathbf{U}$) produces a vector which is called the *curl* of $\mathbf{U}$.

Operation $\mathbf{U} \cdot \nabla$ (**note** that it is different to the divergence $\nabla \cdot \mathbf{U}$) results in a scalar operator, while further applying it to $\mathbf{U}$ produces a vector $(\mathbf{U} \cdot \nabla)\mathbf{U}$. On the other hand, $(\nabla \cdot \mathbf{U})\mathbf{U} = \mathbf{U}(\nabla \cdot \mathbf{U})$ (a vector multiplied by a scalar) is also a vector, but not the same as $(\mathbf{U} \cdot \nabla)\mathbf{U}$. In this sense, operation $\mathbf{U} \cdot \nabla$ is not commutative.

Divergence

Take $\nabla \cdot$ (an arbitrary vector field $\vec{a}$):

$$\nabla \cdot \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z\right)$$

$$\begin{aligned} = \quad & \hat{i}\,\frac{\partial}{\partial x} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \\ + \; & \hat{j}\,\frac{\partial}{\partial y} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \\ + \; & \hat{k}\,\frac{\partial}{\partial z} \cdot \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \end{aligned}$$

Expand out all the terms and recall that,
$\hat{i} \cdot \hat{i} = 1, \quad \hat{i} \cdot \hat{j} = 0, \quad \hat{i} \cdot \hat{k} = 0$, etc. Get:

$$\nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

This is known as the divergence of $\vec{a}$. Note that $\vec{a}$ is a vector and ∇ is a vector operator. But $\nabla \cdot \vec{a}$ is a <u>scalar</u> (much as the dot product between 2 vectors is a scalar).

Curl

Consider an arbitrary vector field $\vec{a}$. See what happens when you take $\nabla \times \vec{a}$:

$$\nabla \times \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z\right)$$

$$\begin{aligned} = \; & \hat{i}\,\frac{\partial}{\partial x} \times \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \\ + \; & \hat{j}\,\frac{\partial}{\partial y} \times \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \\ + \; & \hat{k}\,\frac{\partial}{\partial z} \times \left(\hat{i}\,a_x + \hat{j}\,a_y + \hat{k}\,a_z\right) \end{aligned}$$

Expand out all the terms and recall that, $\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{i} \times \hat{k} = -\hat{j}$, etc. Get:

$$\nabla \times \vec{a} = \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \hat{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \hat{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right).$$

$\nabla \times \vec{a}$ is known as the <u>curl</u> of $\vec{a}$. It is a <u>vector</u>!

Don't want to memorize that nasty formula for $\nabla \times \vec{a}$? No problem! The curl can also be written as a determinant:

$$\nabla \times \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \hat{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \hat{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right).$$

Same result as before.

Differentiation formulas involving ∇

If $\vec{a}$ and $\vec{b}$ are vectors and f is a scalar then,

$$\nabla \cdot \left(\vec{a} + \vec{b}\right) = \nabla \cdot \vec{a} + \nabla \cdot \vec{b},$$

$$\nabla \times \left(\vec{a} + \vec{b}\right) = \nabla \times \vec{a} + \nabla \times \vec{b},$$

$$\nabla \cdot \left(f\vec{a}\right) = f\,(\nabla \cdot \vec{a}) + (\nabla f) \cdot \vec{a},$$

$$\nabla \times \left(f\vec{a}\right) = f\,(\nabla \times \vec{a}) + (\nabla f) \times \vec{a},$$

$$\nabla \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}),$$

$$\nabla \times \left(\vec{a} \times \vec{b}\right) = \vec{a}\,(\nabla \cdot \vec{b}) + (\vec{b} \cdot \nabla)\,\vec{a}$$

$$- \vec{b}\,(\nabla \cdot \vec{a}) - (\vec{a} \cdot \nabla)\,\vec{b},$$

$$\nabla \left(\vec{a} \cdot \vec{b}\right) = (\vec{a} \cdot \nabla)\,\vec{b} + (\vec{b} \cdot \nabla)\,\vec{a}$$

$$+ \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{a}).$$