Lecture 10. September 14, 2016

Topics: Vector magnitude and dot product in rotated Cartesian plane coordinates. Transformations of vector projections. Invariance of the magnitude and scalar product to coordinate rotation. Vector (cross) product.

Reading: Section 1.1 of Holton and Hakim, Section 3 of Fiedler.

1. Vector magnitude and dot product in rotated Cartesian plane coordinates

Consider vector $\vec{\mathrm{b}}$ in a Cartesian, (*X*, *Y*, *Z*), system with orthogonal basis $\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}}$ introduced in Class 9:

$$\vec{\mathrm{b}} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}.$$

Now take another 3-D Cartesian system, we will call it (*X′*, *Y′*, *Z*), rotated in the (*X*, *Y*) plane by angle θ relative to the original system. According to the rules of coordinate conversion considered in Class 4:

$$x' = x\cos\theta + y\sin\theta, \qquad y' = y\cos\theta - x\sin\theta, \qquad z' = z.$$

In the (*X′*, *Y′*, *Z*) system we introduce its own basis $\hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$ (these are new unit vectors in the rotated system), which should possess the same properties in the rotated system as the $\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}}$ basis possesses in the original system. **Note** that $\hat{\mathrm{K}} = \hat{\mathrm{k}}$.

Let us consider projections of $\hat{\mathrm{I}}, \hat{\mathrm{J}}$ unit vectors on coordinate directions *x* and *y* of the original system. Based on geometrical considerations:

$$\hat{\mathrm{I}}_x = \cos\theta,\ \hat{\mathrm{I}}_y = \sin\theta,\ \hat{\mathrm{J}}_x = -\sin\theta,\ \hat{\mathrm{J}}_y = \cos\theta,$$

that is

$$\hat{\mathrm{I}} = \cos\theta\hat{\mathrm{i}} + \sin\theta\hat{\mathrm{j}} \text{ and } \hat{\mathrm{J}} = -\sin\theta\hat{\mathrm{i}} + \cos\theta\hat{\mathrm{j}}.$$

It is possible to check (do it yourself, please) that basis $\hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$ satisfies all requirements for the Cartesian basis (see Class 9) in terms of $\hat{\mathrm{I}}\cdot\hat{\mathrm{I}} = 1$, $\hat{\mathrm{I}}\cdot\hat{\mathrm{J}} = 0$ and so on.

Suppose now that we want to write vector $\vec{\mathrm{b}}$ in the new basis $\hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$, i.e., represent it as

$$\vec{\mathrm{b}} = b_{x'}\hat{\mathrm{I}} + b_{y'}\hat{\mathrm{J}} + b_{z'}\hat{\mathrm{K}}.$$

The x' component of vector $\vec{b}$ in the above expression is defined in a usual way using properties of the dot product (Class 9):

$$b_{x'} = \hat{I}\cdot\vec{b} = b_{x'}\hat{I}\cdot\hat{I} + b_{y'}\hat{I}\cdot\hat{J} + b_{z'}\hat{I}\cdot\hat{K}\,.$$

But on the other hand:

$$\hat{I}\cdot\vec{b} = (\cos\theta\hat{i} + \sin\theta\hat{j})\cdot(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) = b_x\cos\theta + b_y\sin\theta\,,$$

so

$$b_{x'} = b_x\cos\theta + b_y\sin\theta\,.$$

Following analogous procedure, we obtain

$$b_{y'} = -b_x\sin\theta + b_y\cos\theta\,.$$

It was illustrated in Class 7 how similar transformations are applied for rotation of the wind vector.

Using the established relations between the components of vector $\vec{b}$ in systems with $\hat{I}$, $\hat{J}$, $\hat{K}$ and $\hat{i}$, $\hat{j}$, $\hat{k}$ bases, we find that

$$\left|\vec{b}\right|^2 = b^2 = b_{x'}{}^2 + b_{y'}{}^2 + b_{z'}{}^2 = b_x{}^2 + b_y{}^2 + b_z{}^2\,,$$

i.e. the magnitude of vector $\vec{b}$ does not change with rotation of the Cartesian system, or, in other words, the *magnitude of a vector is invariant with respect to the coordinate system rotation*.

One may apply the same consideration to a scalar (dot) product of two vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a}\cdot\vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\cdot(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) = a_xb_x + a_yb_y + a_zb_z\,,$$

$$\vec{a}\cdot\vec{b} = (a_{x'}\hat{I} + a_{y'}\hat{J} + a_{z'}\hat{K})\cdot(b_{x'}\hat{I} + b_{y'}\hat{J} + b_{z'}\hat{K}) = a_{x'}b_{x'} + a_{y'}b_{y'} + a_{z'}b_{z'}\,.$$

Recalling that $a_z = a_{z'}$ and $b_z = b_{z'}$, and using

$$a_{x'} = a_x\cos\theta + a_y\sin\theta\,,\; a_{y'} = -a_x\sin\theta + a_y\cos\theta\;\; b_{x'} = b_x\cos\theta + b_y\sin\theta\,,\; b_{y'} = -b_x\sin\theta + b_y\cos\theta$$

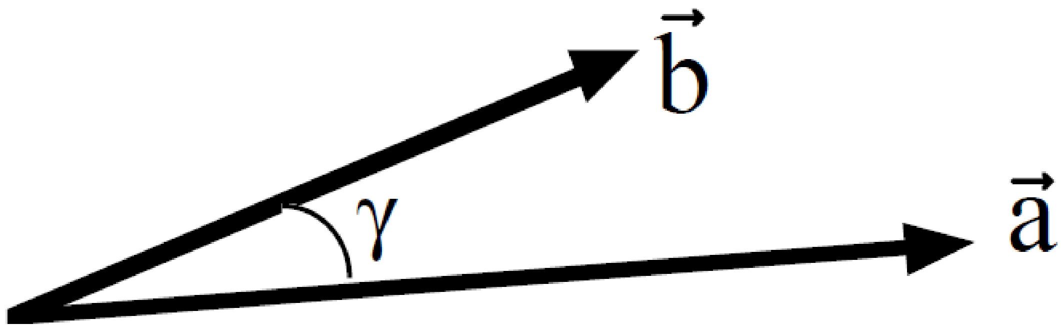
one arrives at

$$\vec{a}\cdot\vec{b} = a_{x'}b_{x'} + a_{y'}b_{y'} + a_{z'}b_{z'} = a_xb_x + a_yb_y + a_zb_z\,,$$

so the scalar product is also invariant with respect to the considered coordinate rotation.

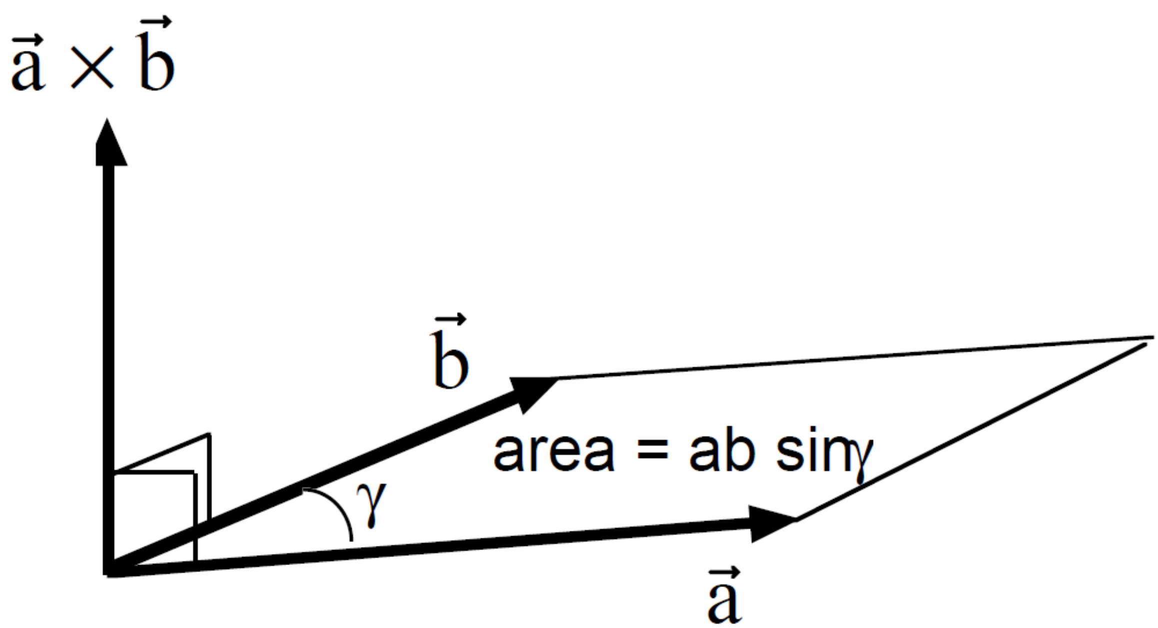
2. Vector (cross) product

Consider two vectors $\vec{a}$ and $\vec{b}$ with angle γ between them (chosen as the smallest of the two angles); a setting analogous to the one used for the dot product consideration in Class 9.



The vector (cross) product of these two vectors is denoted as $\vec{a} \times \vec{b}$ and defined as a *vector* with three properties:

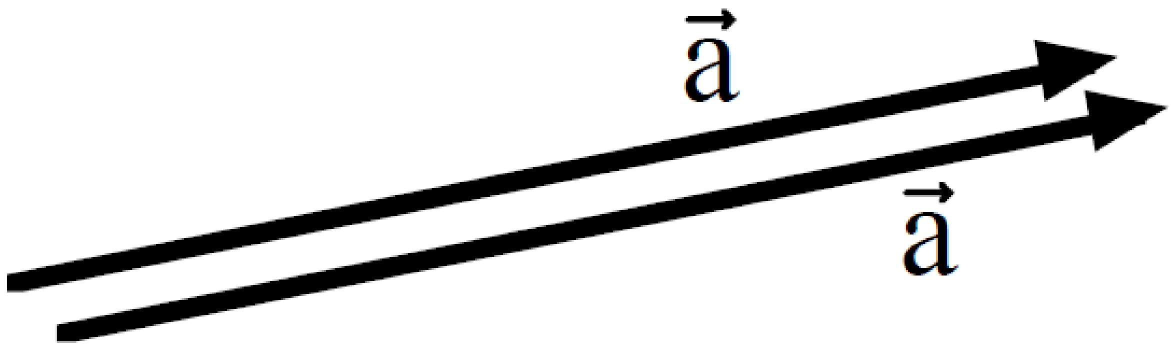
1. Vector $\vec{a} \times \vec{b}$ is normal (perpendicular) to both $\vec{a}$ and $\vec{b}$.
2. Direction of the $\vec{a} \times \vec{b}$ vector is determined by the right-hand rule with fingers curled from $\vec{a}$ toward $\vec{b}$ (this excludes one of possible orientations according to property 1).
3. Magnitude of the $\vec{a} \times \vec{b}$ vector is given by the area of parallelogram formed by vectors $\vec{a}$ and $\vec{b}$, i.e. $|\vec{a} \times \vec{b}| = ab\sin\gamma$, where $a = |\vec{a}|$ and $b = |\vec{b}|$ are, respectively, magnitudes of $\vec{a}$ and $\vec{b}$.



The angle between vectors that results in the largest magnitude of the vector product is $\pi/2$ (90º). **Note** that this value of the angle corresponds to zero value of the dot product (see Class 9).

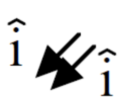
On the other hand, if the two multiplied vectors are parallel, which corresponds to zero or π (90º) angle, the resulting vector product is zero, while the corresponding scaler product is maximal in this case.

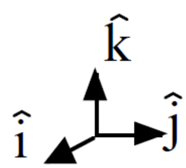
The cross product of vector by itself results in the zero vector because $\sin\gamma = 0$ is this case, so $\vec{a}\times\vec{a}=0$.

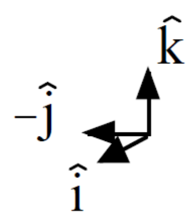


According to the properties of the cross product, its application to the unit basis vectors of Cartesian coordinate system provides:

$$\hat{i}\times\hat{i}=0, \qquad \hat{i}\times\hat{j}=\hat{k}, \qquad \hat{i}\times\hat{k}=-\hat{j}$$







$$\hat{j}\times\hat{i}=-\hat{k}, \qquad \hat{j}\times\hat{j}=0, \qquad \hat{j}\times\hat{k}=\hat{i},$$

$$\hat{k}\times\hat{i}=\hat{j}, \qquad \hat{k}\times\hat{j}=-\hat{i}, \qquad \hat{k}\times\hat{k}=0$$

Vector product $\vec{a}\times\vec{b}$ may be written as the following determinant:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=(a_yb_z-b_ya_z)\hat{i}+(a_zb_x-b_za_x)\hat{j}+(a_xb_y-b_xa_y)\hat{k}\,.$$

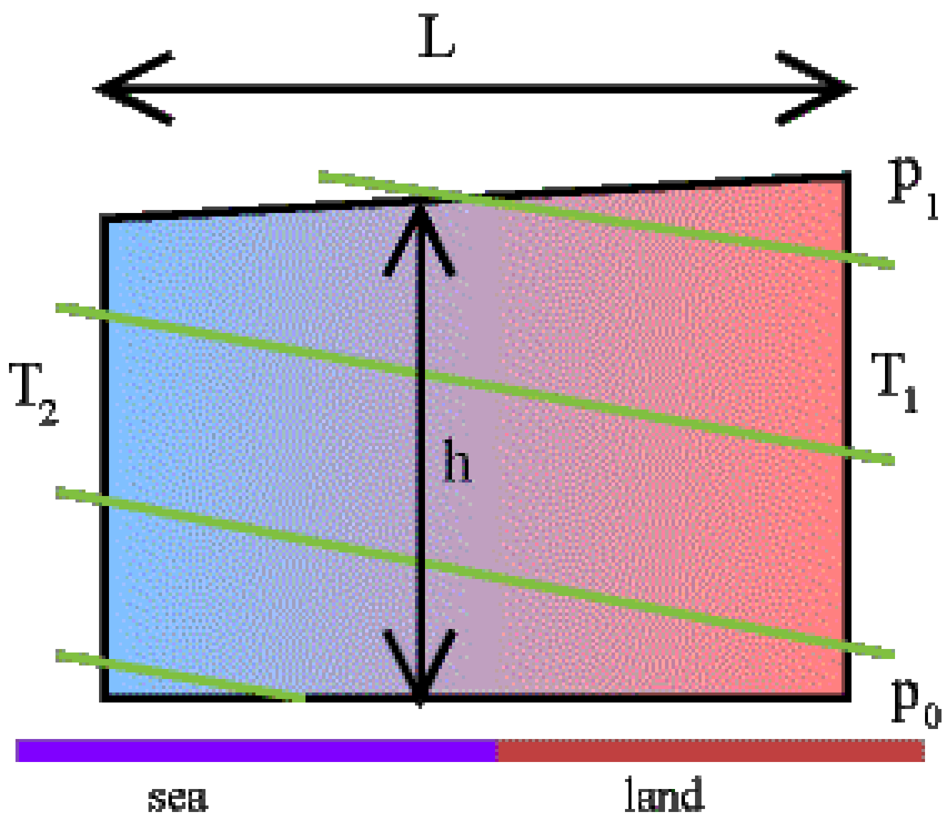
From the right-hand rule and from the determinant representation it follows that the cross product is *not commutative*, that is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=-\vec{b}\times\vec{a}=-\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_x & b_y & b_z\\ a_x & a_y & a_z\end{vmatrix}=-(b_ya_z-a_yb_z)\hat{i}-(b_za_x-a_zb_x)\hat{j}-(b_xa_y-a_xb_y)\hat{k}\,.$$

However, the cross product is *distributive*, so that

$$\vec{c}\times(\vec{a}+\vec{b})=\vec{c}\times\vec{a}+\vec{c}\times\vec{b}\,.$$

Cross Product and the Sea Breeze



The horizontal circulation in the plane shown above is proportional to

$$-\nabla T \times \nabla p$$